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The Effects of Autocorrelation in the Estimation of Process Capability Indices.

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ABSTRACT

The current popularity of the process capability index, a measure of a supplier's ability to meet the product specifications demanded by a customer, has become a matter of some controversy. While admitting the validity of much existing criticism, this research demonstrates that sample estimation of the triple index (*Cpl*, *Cp*, *Cpu*), a variant of the widely used index pair (*Cp*, *Cpk*), is equivalent to estimation of the natural parameters (μ, σ) whenever the measured process characteristic X has an unconditional (marginal) normal probability density function. This includes processes which obey the strictly stationary, normal ARMA(p, q) model. By this extension to stationary normal models beyond ARMA(0, 0), the author shows the continued viability of the process capability index as a decision making tool of wider applicability. Estimators of the indices (*Cpl*, *Cp*, *Cpu*) are studied under conditions of both sample independence and sample autocorrelation. A new method for determining a joint confidence region for the triple index (*Cpl*, *Cp*, *Cpu*) is given. The region presented is, both conceptually and computationally, more direct than previously known approaches.

CHAPTER 1. INTRODUCTION

This work is a critical examination of process capability indices and their sample estimators, in particular, the two most commonly computed single-number indices, *Cp* and *Cpk*. In theory these indices serve as summary measures which aid decision makers in evaluating the capability of a production process. Process capability analysis is the umbrella term for a group of tasks in which these indices, according to some authorities, play too prominent a role.

1.1. Process Capability Analysis

Among the goals of a *process capability analysis*, as summarized by Montgomery (1996), are (1) predicting how well a process will hold its natural tolerances, (2) assisting product developers and designers in selecting or modifying a process, (3) assisting in establishing an interval between sampling for process monitoring, (4) specifying performance requirements for new equipment, (5) selecting between competing vendors, (6) planning the sequence of production processes when there is an interactive effect of the processes on tolerances, and (7) reducing variability in the manufacturing process.

Montgomery (1996) defines process capability analysis as "an engineering study to estimate process capability." This definition, being circular, needs more discussion. For concreteness, suppose a modern, high-volume bottling plant fills two-liter bottles with soft drink. The entire filling process is a complex, highly mechanized operation, designed to run continuously with few stoppages. It is a virtual mathematical certainty that not every bottle filled contains exactly two liters of soft drink. Filled bottles vary in volume. The plant manager knows this, so do his customers, and so do the government agencies charged with monitoring the performance of the bottling plant. Now perhaps the manager reasons that most customers would view an interval from 1.90 to 2.10 liters per bottle as a reasonable *specification interval* and perhaps the government agrees with this reasoning. These *specification limits* may, in some cases, be contractually written between supplier and customer. In this case the government agency may impose monetary fines on the bottler for selling product outside this interval. Certainly in any case, a specification interval serves as a baseline for the manager in the sense that producing within the interval is the minimum goal.

If the plant manager could measure, without error, the quantity of each bottle filled, he would perhaps find an overall pattern or shape to his past census data. He would hope to find that the measured quantities clustered around some target value, say 2.00 liters per bottle. He would be pleased if the very large majority of quantities measured did not vary "too much" around this target, either above or below, say from 1.95 liters to 2.05 liters, and that this variation around the target occurred in a random pattern through time. The manager would describe the process as historically *stable*. Finally, he would hope that this *natural tolerance interval*, from 1.95 to 2.05 liters per bottle for virtually all his bottles, was contained within the specification interval required by his customers. The quality of past stability is important because while no one can see the future, the manager hopes that if conditions remain the same into the future, the past census data, if acceptable, can be repeated. He forecasts the future from the past.

In the more likely case, our plant manager does not measure the quantity of every bottle, but periodically pulls a filled bottle from a shipping carton ready for distribution and measures the volume of soft drink. Without one hundred percent inspection, the manager is now, not only extrapolating future bottle quantities, but interpolating bottle quantities from the past, already filled but unobserved.

Resigning himself to the fact that even in a stable filling process, bottles will vary in quantity over time, the manager will seek to maintain quantities within natural tolerance limits which are themselves within specification limits. Note that in our example, the natural tolerance interval [1.95, 2.05] of the process is completely contained within the specification interval [1.90, 2.10] of the process. The manager refers to his process as *capable*.

Grant and Leavenworth (1988) give five possible courses of action available after comparing natural tolerance limits to specification limits. They are (1) taking no action, when the natural tolerance limits of a process fall well within the specification limits, (2) adjusting the center, when the natural tolerance range is about the same as the specification range, but an adjustment of the center is necessary, (3) reducing variablility, which is usually the more complex action, often requiring changes in methods, tooling, materials, or equipment, (4) changing the specifications, which may be negotiable, and (5) resigning to losses, in which case the focus shifts to scrap and rework costs.

1.2. The Stability of a Process

At the minimum, a process capability analysis requires an attempt by management to use observed past product characteristics in order to both forecast future characteristics and backcast past, but unobserved, characteristics. In other words, by an early stage, a process capability analysis must include a *model* of the measured characteristic X. If X is produced sequentially through time, we are seeking to model $F_{\{X_t\}_{-\infty}^{+\infty}}$, the joint cumulative distribution function of a time series $\{X_t\}_{-\infty}^{+\infty}$.

Of course, obtaining complete knowledge of $F_{\{X_t\}_{-\infty}^{\infty}}$ by observation is not possible. The science of mathematics has a lot to say about computing probabilities when the generating model is assumed. The much more difficult problem is often referred to as inference or *inverse probability*. Given realizations $\{x_1, x_2, \ldots, x_n\}$, what was the joint cdf $F_{\{X_t\}_1^n}$ that generated them? The difficulties are of both aliases and dimensions. There are so many alternative F's which could have spawned this string of numbers. It gets worse. Perhaps the joint cdf F is changing through time, making it difficult or impossible to forecast or backcast. We get a perspective on the enormity of the inference problem when we reflect that observing a realization of a time series $\{x_1, x_2, \ldots, x_n\}$ is really observing a sample of size one. The problem may not be hopeless. If the random variables $\{X_t\}_{-\infty}^{+\infty}$ possess a stationary ergodic structure, in which time averages possess the same information as ensemble averages, it is possible to make inferences on their joint cumulative distribution function F from just one realization through time.

We will take as our definition of process stability, a process such that the unconditional (marginal) cumulative distribution function of X_t neither changes with, nor depends on its time index t. It is as if each X in the time series, unconditional on others in the time series, is drawn from the same marginal cdf F_X.

If $\{X_t\}_{-\infty}^{+\infty}$ are *identically distributed* then by definition, each of any finite number of the X's possesses the same *marginal* cumulative distribution function F_X. *Independent and identically distributed* means that the joint cdf of any finite number of these random variables factors into identical marginals. However, our definition of stability will not demand this factorization. To see this, let $X_t - \mu = \phi(X_{t-1} - \mu) + a_t$, where a_t are independent, identically distributed normal random errors with mean zero and constant finite variance σ_a^2 for all integer t, μ is a finite constant, and ϕ is a constant such that $-1 < \phi < 1$. This is the so-called stationary normal autoregressive process of order one, denoted AR(1). Now for each integer t, X_t is marginally normal with mean μ and variance $\sigma^2 = \sigma_a^2/(1-\phi^2)$ which do not depend on t. So the time-ordered collection $\{X_t\}_{-\infty}^{+\infty}$ is called identically distributed. Yet the joint cdf of any finite number of $\{X_t\}_{-\infty}^{+\infty}$ does not factor into these identical marginals because they are not independent.

We see that the working definition of stability used in the statistical process control literature includes the case of independent, identically distributed characteristics, but is more general. In fact, the so-called strictly stationary time series models qualify as viable stochastic models under our definition of process stability. See Box and Jenkins (1993) or Hamilton (1994) for an extensive survey of these models.

1.3. The Indices *Cp*, *Cpl*, *Cpu*, and *Cpk*

Suppose the bottling process has been running in a stable manner for a long time with a mean μ of 2.00 liters per bottle and that almost all bottles measured inside the natural tolerance interval [*LTL*, *UTL*] = [1.95, 2.05] in liters per bottle. Suppose further that the specification interval is [*LSL*, *USL*] = [1.90, 2.10] in liters per bottle. If we divide the length of the specification interval by the length of the natural tolerance interval, we get

$$\frac{(USL-LSL)}{(UTL-LTL)} = \frac{(2.10-1.90)}{(2.05-1.95)} = \frac{0.20}{0.10} = 2.00.$$

The natural tolerance interval fits inside the specification interval twice. The process is "twice capable." This is good in the sense that the natural variability of the bottling process is only half of what is required by the customer, as measured by the length of the specification interval.

This ratio of specification interval length to natural tolerance interval length is the motivation behind the measure known as the Cp index,

$$Cp = \frac{USL - LSL}{6\sigma}. \tag{1.1}$$

The natural tolerance interval length (UTL - LTL) is denoted 6σ in accordance with the common assumption that the measured characteristic X is normal and so about 99.73 percent (almost all) of the probability of X is within plus or minus three standard deviations σ of the process mean μ.

Now suppose a competitor to our bottler is filling two-liter bottles of soft drink in a stable normal manner, with a natural tolerance interval $[LTL, UTL] = [1.85, 1.90]$, on the same specification interval $[LSL, USL] = [1.90, 2.10]$. He computes his Cp as

$$\frac{(USL - LSL)}{(UTL - LTL)} = \frac{(2.10 - 1.90)}{(1.90 - 1.85)} = \frac{0.20}{0.05} = 4.00.$$

Does this mean the competitor is twice as capable as our bottler with his Cp of 2.00? Of course, it should not. In fact, the competitor is currently filling almost all his product outside specifications, whereas our bottler is filling almost all his product within specifications. The problem, of course, is that while our bottler is centered in the specification interval, the competitor is not. The Cp index ignores the process mean μ. It is not a part of the definition of Cp and does not enter into the calculation of Cp at all. We conclude from this example that the Cp index is not an unambiguous measure of process capability whenever the process mean μ does not fall at the midpoint of the specification interval $m = (LSL + USL)/2$. At most we should interpret the Cp value as a measure of *potential* process capability, conditional on the supplier's ability to center the process mean μ at the midpoint m of the specification interval.

On the other hand, the *Cpk* index was designed for a process which is not centered at the midpoint of the specification interval. First define two indices, a lower index *Cpl* and an upper index *Cpu*. Then take *Cpk* as the minimum of the lower and upper indices, that is,

$$Cpk = \min\{Cpl, Cpu\} = \min\left\{\frac{\mu - LSL}{3\sigma}, \frac{USL - \mu}{3\sigma}\right\}. \tag{1.2}$$

Note that if $\mu = m = (LSL + USL)/2$, then $Cpk = Cp$. Otherwise *Cpk* is strictly less than *Cp*.

We calculate the *Cpk* of our bottler as

$$\begin{aligned} Cpk &= \min\{Cpl, Cpu\} = \min\left\{\frac{\mu - LSL}{3\sigma}, \frac{USL - \mu}{3\sigma}\right\} \\ &= \min\left\{\frac{2.00 - 1.90}{0.05}, \frac{2.10 - 2.00}{0.05}\right\} = \min\{2.00, 2.00\} = 2.00 \end{aligned}$$

and for his competitor as

$$Cpk = \min\left\{\frac{1.875 - 1.90}{0.025}, \frac{2.10 - 1.875}{0.025}\right\} = \min\{-1.00, 9.00\} = -1.00.$$

It appears that the competitor, with his *Cpk* of negative one, is not more capable than our bottler. We will shortly see what this negative one means.

1.4. The Proportion π_0 of Product Outside Specification

Now given that the process characteristic X is stable and normal with mean μ and standard deviation σ, there exists a relationship between the indices (*Cpl*, *Cpu*) and the proportion π_0 of product outside the specification interval. We have

$$\begin{aligned} \pi_0 = \pi_{0l} + \pi_{0u} &= \int_{-\infty}^{LSL} f_X(x)\,dx + \int_{USL}^{\infty} f_X(x)\,dx \\ &= \Phi\left[\frac{LSL - \mu}{\sigma}\right] + 1 - \Phi\left[\frac{USL - \mu}{\sigma}\right] \\ &= \Phi[-3Cpl] + 1 - \Phi[3Cpu] \\ &= \Phi[-3Cpl] + \Phi[-3Cpu], \end{aligned} \tag{1.3}$$

where $f_X(x)$ is the probability density function of a normal random variable with mean μ and standard deviation σ, Φ is the standard normal cumulative distribution function, and (π_{0l}, π_{0u}) are the proportions of X produced below the lower specification limit and above the upper specification limit. If only *Cpk* is known, then we have bounds on π_0 given by

$$\Phi[-3Cpk] < \pi_0 \le 2\Phi[-3Cpk]. \tag{1.4}$$

For the centered process, we have $Cpl = Cp = Cpu$, and so

$$\pi_0 = \Phi[-3Cpl] + \Phi[-3Cpu] = 2\Phi[-3Cp]. \tag{1.5}$$

Note the lack of an exact functional relation between *Cpk* and π_0 except at the boundary defined by $Cpl = Cp = Cpu$. This is because the *Cpk* index "throws away" a piece of information. It throws away either *Cpl* or *Cpu* and we may not know which. To calculate π_0, one needs either (*Cpl*, *Cpu*) or (*Cp*, *Cpk*), since

$$\pi_0 = \Phi[-3Cpl] + \Phi[-3Cpu] = \Phi[-3Cpk] + \Phi[-3(2Cp - Cpk)]. \tag{1.6}$$

In any case, *Cpk* alone does not determine π_0.

We calculate the proportion π_0 of product outside specification for our bottler as

$$\pi_0 = 2\Phi[-3Cp] = 2\Phi[-3(2.00)] = 2\Phi[-6] \approx 0$$

and for his competitor as

$$\pi_0 = \Phi[-3Cpl] + \Phi[-3Cpu] = \Phi[-3(-1.00)] + \Phi[-3(9.00)]$$
$$= \Phi[+3] + \Phi[-27] \approx 0.9986.$$

1.5. The (*Cpl*, *Cpu*) Indices as Reparameterization of (μ, σ)

In a technical sense, calculating and using the indices *Cp* and *Cpk* amounts to a reparameterization of the problem of measuring process capability and it is worthwhile examining explicitly under what conditions this reparameterization is informationally equivalent or invariant.

Consider a process characteristic X which is normally distributed with known mean μ and known standard deviation σ. Let the specification limits be two known constants with LSL strictly less than USL. From the definitions

$$(Cpl, Cp, Cpu) = \left(\frac{\mu - LSL}{3\sigma}, \frac{USL - LSL}{6\sigma}, \frac{USL - \mu}{3\sigma}\right), \tag{1.7}$$

we immediately have the very important relation

$$Cp = \frac{1}{2}(Cpl + Cpu). \tag{1.8}$$

Also, with (USL - LSL) and σ each assumed positive, it follows that

$$Cp = \frac{USL - LSL}{6\sigma} = \frac{1}{2}(Cpl + Cpu) > 0,$$

and so $Cpl + Cpu$ is positive. In the interest of clarity, we will often display the triple index (Cpl, Cp, Cpu), but we must keep in mind that the middle coordinate is always the simple arithmetic mean of the two outer coordinates. The pair (Cpl, Cpu) displays the real action.

Now from

$$(Cpl, Cpu) = \left(\frac{\mu - LSL}{3\sigma}, \frac{USL - \mu}{3\sigma}\right), \tag{1.9}$$

we can solve for (μ, σ) to get

$$(\mu, \sigma) = \left(LSL + \frac{Cpl}{Cpl + Cpu}(USL - LSL), \frac{USL - LSL}{3(Cpl + Cpu)}\right). \tag{1.10}$$

Equations (1.9) or (1.10) define a bijective mapping between two regions of the real plane given by

$$\left\{(\mu, \sigma) \in R^2 \mid \sigma > 0\right\} \quad \text{and} \quad \left\{(Cpl, Cpu) \in R^2 \mid Cpl + Cpu > 0\right\}. \tag{1.11}$$

With this mapping, the cat is out the proverbial bag. And in our introductory chapter, no less. It would appear that there is nothing to be gained from using the (*Cpl*, *Cpu*) parameterization over the (μ, σ) parameterization, at least from the purely technical viewpoint of equivalence. Given fixed *LSL* and *USL*, one can safely go back and forth between two points (μ^*, σ^*) and (C^*pl, C^*pu) and never worry about straying from the path linking them.

We are careful to point out here that while the (*Cpl*, *Cpu*) parameterization is equivalent to the (μ, σ) parameterization, the (*Cp*, *Cpk*) "parameterization" is equivalent to neither. In fact, it is an abuse of language to refer to (*Cp*, *Cpk*) as a parameterization at all, hence the quotation marks. Put simply, one cannot recover either (μ, σ) or (*Cpl*, *Cpu*) from (*Cp*, *Cpk*), even given the specification limits *LSL* and *USL*. We feel it important to state this fact since a casual reading of the practitioner literature would lead one to the conclusion that (*Cp*, *Cpk*) is equivalent to (μ, σ) at all levels of decision making. This is simply not the case. The confusion is partly due to the fact that the proportion π_0 of product outside specification *can* be determined from (*Cp*, *Cpk*), as we have seen, since

$$\pi_0 = \Phi[-3Cpk] + \Phi[-3(2Cp - Cpk)].$$

Now we realize that quality control personnel probably do not "throw out" the *Cpl* and *Cpu* indices when they compute a *Cpk* index. Yet in light of this potential loss of information when *Cpk* is taken as the minimum of *Cpl* and *Cpu*, we recommend that the *Cpk* index be avoided. The two constituent one-sided indices, *Cpl* and *Cpu*, should be reported without confounding. In fact, we believe that it makes good practice to give the triple (*Cpl*, *Cp*, *Cpu*). The middle index is always the simple arithmetic mean of the two outer indices. The *Cpk* index, if desired, can be gotten visually as the minimum of the two outer indices. Note carefully that it is quite possible for the left index *Cpl* to be greater than the right index *Cpu*. We suggest the convention of order (*Cpl, Cp, Cpu*), with *Cpl* on the left, *Cp* in the middle, and *Cpu* on the right. The first and third indices are measures of the actual performance of the process while the middle index is a measure of potential performance conditional on the process mean being adjusted to the center of the specification interval. For example, suppose that one of either *Cpl* or *Cpu* is unacceptable, yet *Cp* is acceptable. This tells us that the process mean μ needs centering at $m = (LSL + USL)/2$, accomplished perhaps with a relatively simple adjustment by an operator. On the other hand, if *Cp* is unacceptable, then the length of the natural tolerance interval as measured by 6σ is unacceptably wide, which is likely to be the more serious case. Its correction could very well involve a major action such as capital investment in equipment.

Returning to our illustration, we see that our bottler has a triple index of

$$(Cpl, Cp, Cpu) = (2.00, \quad 2.00, \quad 2.00).$$

The three coordinates are equal, indicating that he is correctly centered in the specification interval. Further, he is twice capable. The competitor has a triple index of

$$(Cpl, Cp, Cpu) = (-1.00, \quad 4.00, \quad 9.00).$$

If he can center his process without upsetting the spread of his process, he is then four times capable. We can tell that he is not centered in the specification interval because the three coordinates are not equal. In fact, the *Cpl* of negative one indicates that his process mean is below the lower specification limit.

1.6. The (*Cpl, Cp, Cpu*) Diagram

Figure 1.1 introduces a schema which we call the (*Cpl, Cp, Cpu*) *diagram* or, for ease of pronunciation, the *triple index diagram*. It is not meant as a computational tool but rather

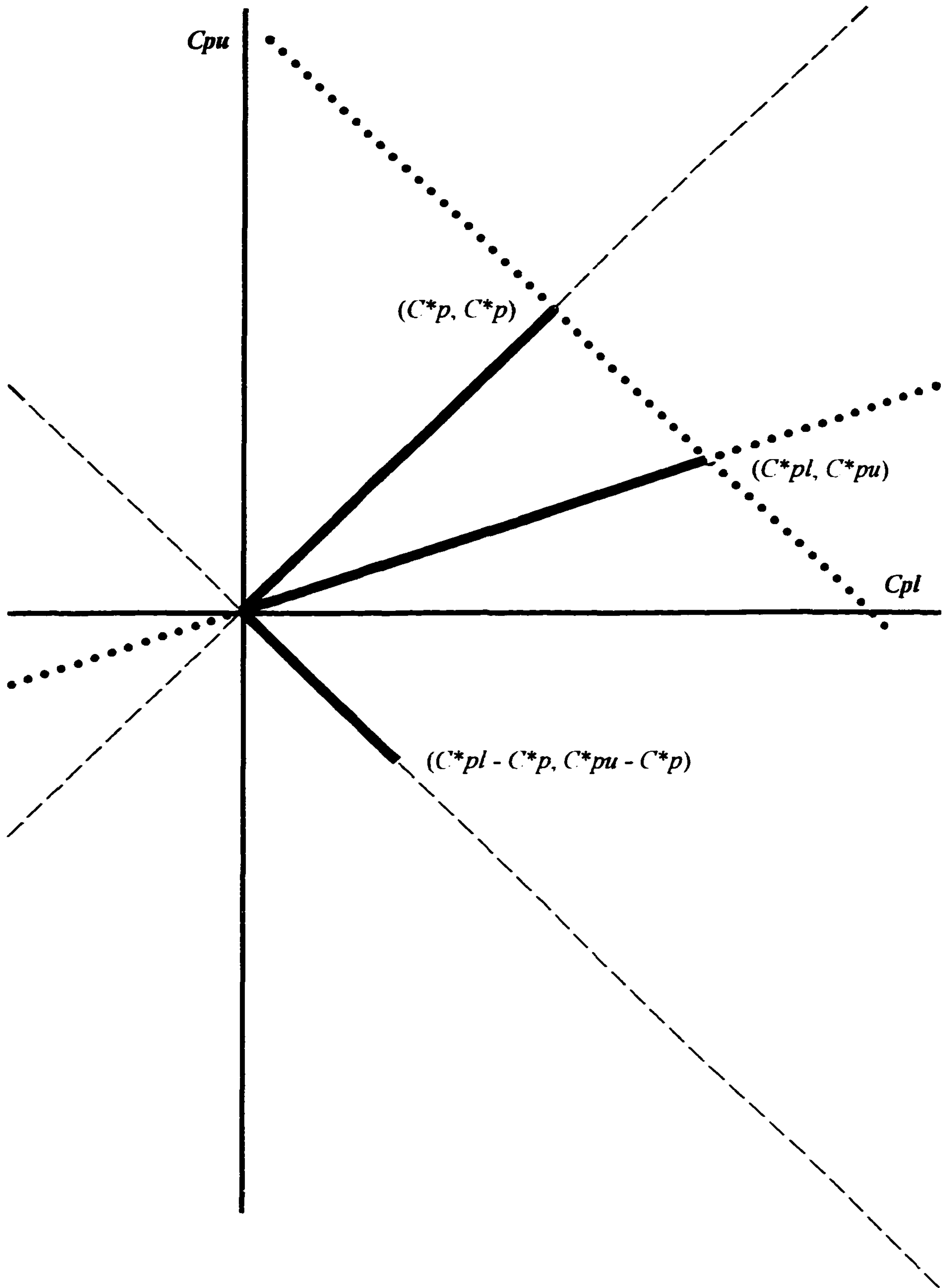

Figure 1.1. The (*Cpl*, *Cp*, *Cpu*) Diagram

as an analytical learning device. With it, we hope to see how the three components of the triple vary in response to changes in the natural parameters (μ, σ). Later we will relate the diagram to the proportion π_0 of product outside specification.

Figure 1.1 plots points (Cpl, Cpu) in the real plane, (Cpl, 0) on the horizontal axis. The two dashed diagonal lines through the origin (0, 0) represent the two lines $Cpl + Cpu = 0$ and $Cpl - Cpu = 0$. Each will be prominent in what follows. We call the line $Cpl + Cpu = 0$, the *boundary of feasiblity*. We call the positive ray of the line $Cpl - Cpu = 0$, the *ray of potentiality*.

With (USL - LSL) and σ each assumed positive, it follows that

$$Cp = \frac{USL - LSL}{6\sigma} = \frac{1}{2}(Cpl + Cpu) > 0,$$

and so $Cpl + Cpu$ is positive. Therefore, the points (Cpl, Cpu) are restricted to lie strictly above the boundary of feasibility, that is, above the line $Cpl + Cpu = 0$. But note that each of Cpl and Cpu can be negative, although not simultaneously.

The orthogonal projection of the fixed point (C^*pl, C^*pu) onto the ray of potentiality (the line $Cpl - Cpu = 0$) is the point (C^*p, C^*p), where $C^*p = (C^*pl + C^*pu)/2$. In this manner, we "recover" the triple index (C^*pl, C^*p, C^*pu) from the point (C^*pl, C^*pu) and its orthogonal projection (C^*p, C^*p). The orthogonal projection of (C^*pl, C^*pu) onto the boundary of feasibility (the line $Cpl + Cpu = 0$) is the point ($C^*pl - C^*p$, $C^*pu - C^*p$). Of course, $(C^*pl, C^*pu) = (C^*p, C^*p) + (C^*pl - C^*p, C^*pu - C^*p)$.

The point (C^*pl, C^*pu) is a measure of the current process capability. Its projection, the point (C^*p, C^*p), is a measure of the potential process capability if, while holding the process standard deviation σ fixed, the process mean μ could be moved to the specification interval midpoint $m = (LSL + USL)/2$.

Given any fixed point (C^*pl, C^*pu) in the feasible region $Cpl + Cpu > 0$, it is *possible* for one, but only one of its two coordinates to be negative. This happens when the mean μ lies outside the specification interval [*LSL*, *USL*]. Given any fixed point (C^*pl, C^*pu) not on the ray of potentiality, the point ($C^*pl - C^*p$, $C^*pu - C^*p$) will have one negative and one positive coordinate. Further, the two coordinates sum to zero, being deviations from a mean. Of course, the point (C^*p, C^*p) must lie in the positive quadrant, on the ray of potentiality.

The dotted line through the point (C^*pl, C^*pu), parallel to the boundary of feasibility and perpendicular to the ray of potentiality, is the line $Cpl + Cpu = C^*pl + C^*pu$. It is the trace of points (Cpl, Cpu), starting at the point (C^*pl, C^*pu), as μ varies through the reals while σ remains fixed. On the other hand, the dotted line through the point (C^*pl, C^*pu) and the origin (0, 0) is the line $Cpu/Cpl = C^*pu/C^*pl$. Its feasible portion is the trace of points (Cpl, Cpu), starting at the point (C^*pl, C^*pu), as σ varies through the positive reals while μ remains fixed.

1.7. The (*Cpl*, *Cpu*) Indices as Measure of Actual versus Ideal (μ, σ)

Consider a Supplier A and his process characteristic X which is normally distributed with mean μ_A and standard deviation σ_A. We have already talked of $[LTL, UTL] = [A_0, A_1]$, the *actual* natural tolerance interval of a normal process, and its relation to the process parameters

$$(\mu_A, \sigma_A) = \left(\frac{1}{2}(A_0 + A_1),\ \ \frac{1}{6}(A_1 - A_0)\right). \tag{1.12}$$

Now when Customer I gives a specification interval $[LSL, USL] = [I_0, I_1]$ to Supplier A, he is, in a sense, communicating to the supplier an *ideal* random variable which we take to be normal with mean μ_I and standard deviation σ_I such that

$$(\mu_I, \sigma_I) = \left(\frac{1}{2}(I_0 + I_1), \quad \frac{1}{6}(I_1 - I_0)\right). \tag{1.13}$$

We have

$$Cp = \frac{USL - LSL}{6\sigma} = \frac{I_1 - I_0}{A_1 - A_0} = \frac{(I_1 - I_0)/6}{(A_1 - A_0)/6} = \frac{\sigma_I}{\sigma_A}. \tag{1.14}$$

In other words, the Cp index can be interpreted as the ratio of the customer's ideal process standard deviation to the supplier's actual process standard deviation. This is a noise-to-noise ratio. If Cp is less than one, then the supplier's currently attainable noise level is greater than the customer's allowable noise level. Continuing, we have

$$(\mu_I, \sigma_I) = \left(\frac{1}{2}(I_0 + I_1), \quad \frac{1}{6}(I_1 - I_0)\right),$$

implying

$$(I_0, I_1) = (\mu_I - 3\sigma_I, \quad \mu_I + 3\sigma_I), \tag{1.15}$$

yielding

$$\begin{aligned}
(Cpl, Cpu) &= \left(\frac{\mu - LSL}{3\sigma}, \quad \frac{USL - \mu}{3\sigma}\right) = \left(\frac{\mu_A - I_0}{3\sigma_A}, \quad \frac{I_1 - \mu_A}{3\sigma_A}\right) \\
&= \left(\frac{\mu_A - (\mu_I - 3\sigma_I)}{3\sigma_A}, \quad \frac{(\mu_I + 3\sigma_I) - \mu_A}{3\sigma_A}\right) \\
&= \left(\frac{\sigma_I}{\sigma_A} + \frac{\mu_A - \mu_I}{3\sigma_A}, \quad \frac{\sigma_I}{\sigma_A} + \frac{\mu_I - \mu_A}{3\sigma_A}\right) \\
&= \left(Cp + \frac{\mu_A - \mu_I}{3\sigma_A}, \quad Cp + \frac{\mu_I - \mu_A}{3\sigma_A}\right).
\end{aligned} \tag{1.16}$$

Since $Cp = (Cpl + Cpu)/2$ is a noise-to-noise ratio, then each of Cpl and Cpu is a noise-to-noise ratio, although Cpl and Cpu are perturbed. This is because all three have denominators in the parameter σ_A only. We could choose to display the triple index (Cpl, Cp, Cpu), in terms of the actual and ideal natural parameters, as

$$(Cpl, Cp, Cpu) = \left(\frac{\sigma_I}{\sigma_A} + \frac{\mu_A - \mu_I}{3\sigma_A},\ \frac{\sigma_I}{\sigma_A},\ \frac{\sigma_I}{\sigma_A} + \frac{\mu_I - \mu_A}{3\sigma_A} \right). \tag{1.17}$$

If the supplier is centered in the specification interval, then $\mu_A = \mu_I$ and so

$$(Cpl, Cp, Cpu) = (Cp, Cp, Cp).$$

1.8. A Compendium of Bijections

We bring together several bijections for further insight. In order to maintain a consistent notation, let us refer to Supplier A, where the letter "A" is meant to suggest "actual" throughout this section. We also refer to Customer I, where the letter "I" is meant to suggest "ideal." Let us denote the supplier's *actual* natural tolerance interval $[LTL, UTL]$ as $[A_0, A_1]$. Let us denote the customer's *ideal* specification interval $[LSL, USL]$ as $[I_0, I_1]$. We then have for Supplier A,

$$(\mu_A, \sigma_A) = \left(\frac{1}{2}(A_0 + A_1),\ \frac{1}{6}(A_1 - A_0) \right)$$

$$\text{or} \quad (A_0, A_1) = (\mu_A - 3\sigma_A,\ \ \mu_A + 3\sigma_A),$$

which define the supplier's bijection between the regions

$$\left\{ (\mu_A, \sigma_A) \in R^2 \,|\, \sigma_A > 0 \right\} \quad \text{and} \quad \left\{ (A_0, A_1) \in R^2 \,|\, A_0 < A_1 \right\}.$$

For Customer I, we have

$$(\mu_I, \sigma_I) = \left(\frac{1}{2}(I_0 + I_1),\ \frac{1}{6}(I_1 - I_0) \right)$$

$$\text{or} \quad (I_0, I_1) = (\mu_I - 3\sigma_I,\ \ \mu_I + 3\sigma_I),$$

which define the customer's bijection between the regions

$$\left\{ (\mu_I, \sigma_I) \in R^2 \,|\, \sigma_I > 0 \right\} \quad \text{and} \quad \left\{ (I_0, I_1) \in R^2 \,|\, I_0 < I_1 \right\}.$$

Now the utility of the two indices (Cpl, Cpu) is in *defining a crosslink between the two bijections*. In terms of interval endpoints, we have

$$(Cpl, Cpu) = \left(\frac{(A_0 + A_1) - 2I_0}{A_1 - A_0}, \ \frac{2I_1 - (A_0 + A_1)}{A_1 - A_0} \right),$$

which, for fixed $[I_0, I_1]$, is itself a bijection between the regions of the real plane given by

$$\left\{ (A_0, A_1) \in R^2 \mid A_0 < A_1 \right\} \quad \text{and} \quad \left\{ (Cpl, Cpu) \in R^2 \mid Cpl + Cpu > 0 \right\}.$$

In terms of noise-to-noise ratios, we have

$$(Cpl, Cpu) = \left(\frac{\sigma_I}{\sigma_A} + \frac{\mu_A - \mu_I}{3\sigma_A}, \ \frac{\sigma_I}{\sigma_A} + \frac{\mu_I - \mu_A}{3\sigma_A} \right),$$

which, for fixed (μ_I, σ_I), is itself a bijection between the regions of the real plane given by

$$\left\{ (\mu_A, \sigma_A) \in R^2 \mid \sigma_A > 0 \right\} \quad \text{and} \quad \left\{ (Cpl, Cpu) \in R^2 \mid Cpl + Cpu > 0 \right\}.$$

We could choose to display the triple index

$$(Cpl, Cp, Cpu) = \left(\frac{\mu - LSL}{3\sigma}, \ \frac{USL - LSL}{6\sigma}, \ \frac{USL - \mu}{3\sigma} \right)$$

$$= \left(\frac{(A_0 + A_1) - 2I_0}{A_1 - A_0}, \ \frac{I_1 - I_0}{A_1 - A_0}, \ \frac{2I_1 - (A_0 + A_1)}{A_1 - A_0} \right)$$

$$= \left(\frac{\sigma_I}{\sigma_A} + \frac{\mu_A - \mu_I}{3\sigma_A}, \ \frac{\sigma_I}{\sigma_A}, \ \frac{\sigma_I}{\sigma_A} + \frac{\mu_I - \mu_A}{3\sigma_A} \right).$$

Note that $Cp = (Cpl + Cpu)/2$, as it should.

1.9. Criticisms of the Indices *Cp*, *Cpl*, *Cpu*, and *Cpk*

These facts no doubt prompted M. Johnson (1992) to assert that none of the process capability indices *Cp*, *Cpl*, *Cpu*, or *Cpk* adds any knowledge or understanding beyond that contained in the basic parameters mean μ, standard deviation σ, target value m, and specification limits [*LSL*, *USL*]. While this is a criticism at the most basic level, yet we feel it to be the easiest to answer. His argument is that since the indices are the result of a reparameterization of μ and σ and nothing more, they are therefore unnecessary. But "unnecessary" is not the same thing as "useless." This is like saying that the coefficient of variation σ/μ is "unnecessary" because it adds no knowledge beyond that contained in the basic parameters μ and σ. Yet decision makers find the coefficient of variation useful in comparing two normal distributions, which is exactly what is going on with the process capability indices. As another example, in Bayesian statistics it is much easier to work with the precision parameter rather than the variance parameter of the normal random variable. The precision is defined as the inverse of the variance, that is, $\gamma = 1/\sigma^2$. Now it is not necessary to do this, but it is efficient. It is ultimately a question of utility versus cost.

The indices (Cpl, Cp, Cpu) were designed by engineers who saw the problem in terms of noise-to-noise and signal-to-noise ratios. It is not surprising to find that statisticians without engineering background disparage this index approach and prefer working with the natural parameters (μ, σ). We take a middle ground. It is our view that, barring misspecification of the probability law governing the process X, the indices (Cpl, Cp, Cpu) have an *interpretive* role in aiding decision makers. However, we also feel that the natural parameters (μ, σ) must suggest the *estimative* approach to be taken.

Another criticism of process capability indices is that they are computed using sample estimates $(\hat{\mu}, \hat{\sigma})$ of the population parameters (μ, σ). This results in sample estimates $(\hat{C}pl, \hat{C}p, \hat{C}pu)$ of the population indices (Cpl, Cp, Cpu). For example, given the specification limits $[LSL, USL]$ and the natural moment estimates

$$(\hat{\mu}, \hat{\sigma}) = (\bar{x}, s) = \left(\frac{1}{n}\sum_{i=1}^{n} x_i,\ \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2} \right), \tag{1.18}$$

one could compute natural estimates

$$(\hat{C}pl, \hat{C}p, \hat{C}pu) = \left(\frac{\hat{\mu} - LSL}{3\hat{\sigma}},\ \frac{USL - LSL}{6\hat{\sigma}},\ \frac{USL - \hat{\mu}}{3\hat{\sigma}} \right)$$

$$= \left(\frac{\bar{x} - LSL}{3s}, \ \frac{USL - LSL}{6s}, \ \frac{USL - \bar{x}}{3s} \right). \tag{1.19}$$

Often these point estimates are reported without any indication of their sampling variability. This criticism is surely a valid one. However, in light of the equivalence of the parameterizations, it follows that much of the criticism directed at the use of the capability indices (*Cpl*, *Cp*, *Cpu*) is not about the indices at all, but is about the absence of inferential procedure in general. These problems would still exist even using the natural parameterization (μ, σ), which is to say, the sampling variability in sample estimators of (μ, σ) can be ignored quite as simply as the sampling variability in the estimators of (*Cpl*, *Cp*, *Cpu*) can be ignored. It is a fact that given *LSL* and *USL*, X can be parameterized in (*Cpl*, *Cpu*) as in (μ, σ).

Furthermore, finding an estimator with good sampling properties for the pair of indices (*Cpl*, *Cpu*) is practically equivalent, which is to say, only a little more arduous, than finding a good estimator for the pair (μ, σ). To see this, note that the joint maximum likelihood estimators of (μ, σ) from an independent, identically normal sample of size n are given by

$$(\hat{\mu}_{ML}, \hat{\sigma}_{ML}) = \left(\frac{1}{n}\sum_{i=1}^{n} X_i, \ \sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2} \right). \tag{1.20}$$

It follows from the invariance property of maximum likelihood estimators under regularity conditions that the joint maximum likelihood estimators of (*Cpl*, *Cpu*) are

$$\left(\hat{C}pl_{ML}, \hat{C}pu_{ML}\right) = \left(\frac{\hat{\mu}_{ML} - LSL}{3\hat{\sigma}_{ML}}, \ \frac{USL - \hat{\mu}_{ML}}{3\hat{\sigma}_{ML}} \right), \tag{1.21}$$

while the joint maximum likelihood estimators of (π_{0l}, π_{0u}) are

$$\left(\hat{\pi}_{0l_{ML}}, \hat{\pi}_{0u_{ML}}\right) = \left(\Phi\left[-3\hat{C}pl_{ML}\right], \ \Phi\left[-3\hat{C}pu_{ML}\right]\right). \tag{1.22}$$

Finally, the maximum likelihood estimator of $Cp = (USL - LSL)/6\sigma$ is given by

$$\hat{C}p_{ML} = \frac{USL - LSL}{6\hat{\sigma}_{ML}}. \tag{1.23}$$

We could choose to display the joint maximum likelihood estimator of (Cpl, Cp, Cpu) as

$$\left(\hat{C}pl_{ML}, \hat{C}p_{ML}, \hat{C}pu_{ML}\right) = \left(\frac{\hat{\mu}_{ML} - LSL}{3\hat{\sigma}_{ML}}, \quad \frac{USL - LSL}{6\hat{\sigma}_{ML}}, \quad \frac{USL - \hat{\mu}_{ML}}{3\hat{\sigma}_{ML}}\right), \tag{1.24}$$

if we remember it to be a two-dimensional random variable.

To take another point estimation strategy, consider uniformly minimum variance unbiased (UMVU) estimators, that is, unbiased estimators which have minimum variance within the class of all unbiased estimators for a particular parameter. It is well known that for an independent, identically normal sample of size n, a complete sufficient statistic for (μ, σ) is given by

$$\left(\bar{X}, S\right) = \left(\frac{1}{n}\sum_{i=1}^{n} X_i, \quad \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2}\right). \tag{1.25}$$

By the theorem of Lehmann and Scheffé, every bijective transformation of $\left(\bar{X}, S\right)$ is also a complete sufficient statistic for (μ, σ). Furthermore, that transformation which is unbiased for (μ, σ) is the unique UMVU estimator of (μ, σ). This UMVU estimator is found to be

$$\begin{aligned}
\left(\hat{\mu}_{UMVU}, \hat{\sigma}_{UMVU}\right) &= \left(\frac{1}{n}\sum_{i=1}^{n} X_i, \quad \sqrt{\frac{n-1}{2}}\frac{\Gamma[(n-1)/2]}{\Gamma[n/2]}\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2}\right) \\
&= \left(\bar{X}, \quad \sqrt{\frac{n-1}{2}}\frac{\Gamma[(n-1)/2]}{\Gamma[n/2]}S\right),
\end{aligned} \tag{1.26}$$

where the gamma function is defined by

$$\Gamma[x] = \int_0^{\infty} t^{x-1}e^{-t}dt,$$

the integral being finite for all real x except the nonpositive integers.

Now since $\left(\frac{\bar{X}-LSL}{3S}, \frac{USL-\bar{X}}{3S}\right)$ is a bijective transformation of $(\bar{X}, S)$, it is itself a complete sufficient statistic for (μ, σ), and one need only find its unbiased multiple to locate the unique UMVU estimator of $(Cpl, Cpu) = \left(\frac{\mu - LSL}{3\sigma}, \frac{USL-\mu}{3\sigma}\right)$. Taking

$$\hat{C}pl_{UMVU} = \frac{\Gamma[(n-1)/2]}{\Gamma[(n-2)/2]}\sqrt{\frac{2}{n-1}}\left(\frac{\bar{X}-LSL}{3S}\right), \tag{1.27}$$

we find

$$E\left[\hat{C}pl_{UMVU}\right] = \frac{\mu - LSL}{3\sigma} = Cpl. \tag{1.28}$$

Also, taking

$$\hat{C}pu_{UMVU} = \frac{\Gamma[(n-1)/2]}{\Gamma[(n-2)/2]}\sqrt{\frac{2}{n-1}}\left(\frac{USL-\bar{X}}{3S}\right), \tag{1.29}$$

we have

$$E\left[\hat{C}pu_{UMVU}\right] = \frac{USL-\mu}{3\sigma} = Cpu. \tag{1.30}$$

By the theorem of Lehmann and Scheffé, $\left(\hat{C}pl_{UMVU}, \hat{C}pu_{UMVU}\right)$ is the unique UMVU estimator of (Cpl, Cpu). Similarly, the unique UMVU estimator of $Cp = (USL - LSL)/6\sigma$ is given by

$$\hat{C}p_{UMVU} = \frac{\Gamma[(n-1)/2]}{\Gamma[(n-2)/2]}\sqrt{\frac{2}{n-1}}\left(\frac{USL-LSL}{6S}\right), \tag{1.31}$$

since

$$E\left[\hat{C}p_{UMVU}\right] = \frac{USL-LSL}{6\sigma} = Cp. \tag{1.32}$$

We could choose to display the unique UMVU estimator of (Cpl, Cp, Cpu) as

$$\left(\hat{C}pl_{UMVU}, \hat{C}p_{UMVU}, \hat{C}pu_{UMVU}\right), \tag{1.33}$$

if we remember that it is a two-dimensional random variable.

A final criticism of the process capability indices reasons that each is a ratio of lengths rather than a direct measure of the proportion π_0 of product outside specification. A simple rebuttal to this attack would begin by observing that so too are the natural parameters (μ, σ) merely a length (from the origin) and a length (from μ to the point of inflection of the density), and not direct measures of π_0.

1.10. Contribution of the Research

The process capability index pair (Cp, Cpk) is widely used in industry. Its study has been the preoccupation of a number of researchers for some years. We wish to shift the focus to the triple index (Cpl, Cp, Cpu), a variant of (Cp, Cpk). One advantage to this shift will be the maintenance of a bijective parametric equivalence with the natural parameter pair (μ, σ) for normal process characteristic X. A second advantage will be an easier method for constructing joint confidence regions in (Cpl, Cp, Cpu) and the proportion π_0 of product outside specification. We are convinced that the Cpk index represents a wayward path, as witnessed by the difficult forms that its current derived body of inferential procedure has revealed.

While the sampling properties of the common estimators for the indices (Cp, Cpk) have been examined for the case of independent, identically distributed normal characteristics, their sampling properties under more general conditions have not. In particular, we investigate estimators of (Cpl, Cp, Cpu) when the sample observations are normal, but autocorrelated. We give a lower bound on the mean of the random variable $\hat{Cp}/Cp$, showing the potential dangers lurking in small samples from autocorrelated processes. This case would include sampling from stationary normal ARMA(p, q) processes, as presented in Box and Jenkins

(1993). In light of the many data sets taken from business and industry which Box and Jenkins analyze, it follows that the sampling properties of capability indices under autocorrelation mark an important area for investigation.

1.11. Organization of the Research

This research is organized into five chapters. Following a first introductory chapter, Chapter 2 presents a review of the relevant background literature which provided the groundwork and motivation for the current research. In Chapter 3, some common estimators of the process capability indices (Cpl, Cp, Cpu) are analyzed with respect to their sampling properties in the simplest case of independent, identically distributed normal measurements. A discussion of why normality is often manifested in real world data is provided. We continue with classical interval estimation, presenting a method for determining a joint confidence interval for the true triple index (Cpl, Cp, Cpu) which is, both conceptually and computationally, more direct than any method previously published. This procedure leads to both point and interval estimators of the proportion π_0 of product outside specification, a parameter which many experts feel to be of foremost importance in process capability analysis.

Chapter 4 begins with a discussion of continuous linear stochastic differential equation models and their relation to discrete linear stochastic difference equation models. In particular, we seek an explanation for the frequent observance of autocorrelation in measured data. After demonstrating a lower bound on the mean of the random variable $\hat{C}p/Cp$, we then take up the sampling properties of estimators of the indices (Cpl, Cp, Cpu) under autocorrelation. This very important area has been largely neglected due to the high degree of mathematical intractability in the problem. Chapter 5 contains our summary and conclusions.

CHAPTER 2. REVIEW OF THE LITERATURE

Process capability indices (PCIs) have been popular for over twenty-five years, since the capability ratio (CR) was introduced by Ekvall and Juran (1974). Variant forms of the CR index have proliferated both in use and variety during the last decade, which has seen the birth of the indices *Cp*, *Cpk*, *Cpm*, *Ppk*, and others. This has sparked significant controversy. Some believe PCIs should be discontinued while others feel they have use in conjunction with other measures. Some use PCIs as absolute measures. Many feel these measures have had a major negative economic impact on industry.

While the process capability indices have led to some improvements, they have, almost certainly, been the cause of many unjust decisions. Procedures for determining process capability by a single index were propagated mainly by over-zealous customers who viewed them as a panacea for problems of quality improvement. Rigid adherence to rules for calculating the indices *Cp* and *Cpk* on a daily basis, with the goal of raising them above 1.333 as much as possible, caused a revolt among a number of influential and open-minded quality control statisticians. Statistical terrorism, unscrupulous manipulation or doctoring, and calls for their elimination are all reported in Kitsa (1991). More moderate voices (Gunter, 1989), and more defenders (McCormick, 1989), (Steenburgh, 1991), (McCoy, 1991), have been heard. This heated debate, which flared in 1991, says something may be wrong with these indices or their use.

Additional indices, introduced by Chan *et al.* (1988), Spiring (1991), Boyles (1991), and Pearn *et al.* (1992), take account of a target not at the specification midpoint or possible non-normality of the original process characteristic. Confusion among practitioners occurs since they have been denied a clear explanation of the meaning and underlying assumptions. This is exacerbated even further because of the twin modern approaches to the PCI, as insensitive measure of nonconforming product versus loss considerations.

The form of the loss function by far the most favored is a quadratic function of μ, but little supporting evidence for this choice has appeared. In fact, the father of squared error loss, W.F. Gauss (1821), defends his choice strictly as a matter of mathematical simplicity and convenience. Should someone object to his specification as arbitrary, he writes, he is in complete agreement. Furthermore, in the context of PCIs, it can intelligently be asked what is gained from this confounding. If one is really interested in a loss function, why not just estimate the expected loss $E[L]$ and not some unnecessarily complicated function of it?

According to Kotz and Johnson (1993), the issue does not lie in their mathematical validity, but in their application by those who believe the values are deterministic rather than stochastic. They feel that once the variability is understood and the bias is known, the use of these PCIs can be more constructive. In fact, Kotz and Johnson advocate process improvement in general, rather than focusing on a single measure or index.

2.1. The *Cp* Index

The Cp index is defined by

$$Cp = \frac{USL - LSL}{6\sigma}, \tag{2.1}$$

where $\sigma^2 = E\left[(X-\mu)^2\right]$. It is, at best, an indirect measure of the potential ability to meet the specification requirement, $LSL \le X \le USL$. Clearly, large values of Cp are desirable and small values undesirable.

What motivates the "6" in the denominator? If X is normal with mean μ and standard deviation σ and $\mu = (LSL + USL)/2$, then the proportion of product outside the specification limits is $2\Phi[-d/\sigma]$, where $d = (USL - LSL)/2$ is the half-width of the specification interval and Φ is the standard normal cumulative distribution function. Since $Cp = d/3\sigma$, $2\Phi[-3Cp]$ is the proportion of product outside the specification limits. If $Cp = 1.00$, then $2\Phi[-3Cp] = 0.27\%$ NC (nonconforming product) or 2700 NCPPM (nonconforming parts per million). Similarly, a Cp of 1.33 gives 63 NCPPM and a Cp of 2.00 gives 0.002 NCPPM. Of course, having a Cp of 1.00 does not guarantee 0.27% NC. There will never be less than 0.27% NC only at $\mu = (LSL + USL)/2$. In other words, a Cp of 1.00 is an indication that it is possible to have NC as small as 0.27%, provided that μ is at the specification interval midpoint.

Carr (1991) believes that in the academic analyses of PCIs, the original motivation has been lost. He suggests simply using NCPPM as a capability index. Constable and Hobbs (1992) define "capable" as referring to percentage of output within specification. Lam and Littig (1992) suggest $3Cpp = \Phi^{-1}\left[(\pi_0 + 1)/2\right]$ and use $3\hat{C}pp = \Phi^{-1}\left[(\hat{\pi}_0 + 1)/2\right]$ based on an estimator of π_0 from the observed X. Wierda (1992) suggests using $-\Phi^{-1}\left[\hat{\pi}_0\right]/3$.

Herman (1989) distinguishes "mechanical industries," such as the automotive industry where the Cp began, and "process industries." The difference is in measurement error. The sample variance $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$ is an estimator of σ^2 + (measurement error variance). In addition, each of these components could have within-lot and between-lot components. He suggests the "process performance index" or PPI,

$$P_p = \frac{USL - LSL}{6\sigma_{TOTAL}}. \tag{2.2}$$

Montgomery (1985) recommends a minimum Cp of 1.33 for existing processes, 1.50 for new processes. For projects of essential safety, 1.50 for existing and 1.67 for new. He does not explain his exact rationale for these numbers, however.

From Kotz and Johnson (1993), the natural moment estimator of π_0 is

$$\hat{\pi}_0 = 1 - \Phi\left[\frac{USL - \bar{X}}{S}\right] + \Phi\left[\frac{LSL - \bar{X}}{S}\right], \tag{2.3}$$

a biased estimator. A UMVU estimator of π_0 exists (Kotz and Johnson, 1993) but is complicated. Each seems to depend on the assumption of normality.

The only parameter in $(USL - LSL)/6\sigma$ is σ. Take $\hat{\sigma} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2}$, where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. If X is $N(\mu, \sigma^2)$, then $\frac{(n-1)\hat{\sigma}^2}{\sigma^2}$ is χ^2_{n-1}. A $100(1 - \alpha)\%$ confidence interval for σ^2 is given by

$$\left[\frac{(n-1)\hat{\sigma}^2}{\chi^2_{n-1,\,1-\alpha/2}}, \frac{(n-1)\hat{\sigma}^2}{\chi^2_{n-1,\,\alpha/2}}\right], \tag{2.4}$$

and so a $100(1 - \alpha)\%$ confidence interval for Cp is

$$\left[\hat{Cp}\sqrt{\frac{\chi^2_{n-1,\,\alpha/2}}{n-1}}, \hat{Cp}\sqrt{\frac{\chi^2_{n-1,\,1-\alpha/2}}{n-1}}\right]. \tag{2.5}$$

Kotz and Johnson give five pages to the approximation of χ^2.

2.2. The *Cpk* Index

The Cpk index was introduced to give μ some influence on the value of the PCI. It is given by

$$Cpk = \min\{Cpl, Cpu\} = \min\left\{\frac{\mu - LSL}{3\sigma}, \frac{USL - \mu}{3\sigma}\right\}$$

$$= \frac{d - \left| \mu - \frac{1}{2}(LSL + USL) \right|}{3\sigma}$$

$$= \left\{ 1 - \frac{\left| \mu - \frac{1}{2}(LSL + USL) \right|}{d} \right\} Cp, \tag{2.6}$$

where $d = (USL - LSL)/2$ is the half-width of the specification interval. Since $Cp = d/3\sigma$, we have $Cpk \leq Cp$, with equality if and only the process mean μ falls at the specification interval midpoint $m = (LSL + USL)/2$. If X is $N(\mu, \sigma^2)$, then the nonconforming product π_0 is

$$\Phi\left[\frac{LSL - \mu}{\sigma}\right] + 1 - \Phi\left[\frac{USL - \mu}{\sigma}\right] = \Phi[-3(2Cp - Cpk)] + \Phi[-3Cpk], \tag{2.7}$$

and therefore,

$$\Phi[-3Cpk] < \pi_0 \leq 2\Phi[-3Cpk]. \tag{2.8}$$

From Kotz and Johnson (1993), take

$$\hat{C}pk = \frac{d - \left| \bar{X} - \frac{1}{2}(LSL + USL) \right|}{3\hat{\sigma}}, \tag{2.9}$$

where $\hat{\sigma} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2}$. Then with $\{X_i\}_n$ *iid* $N(\mu, \sigma^2)$, we have $\bar{X} \sim N(\mu, \sigma/\sqrt{n})$ and $\hat{\sigma} \sim \chi_{n-1}\sigma/\sqrt{n}$, and they are independent. The statistic $\sqrt{n}\left| \bar{X} - \frac{1}{2}(LSL + USL) \right| / \hat{\sigma}$ has a folded normal distribution. It is distributed as $|U + \delta|$, where $U \sim N(0, 1)$ and $\delta = \frac{\sqrt{n}}{\sigma}\left| \mu - \frac{1}{2}(LSL + USL) \right|$, making $E[\hat{C}pk]$ and $Var[\hat{C}pk]$ complicated functions of the parameters (n, μ, σ, LSL, USL). This estimator of Cpk is biased with both positive and negative components. The bias is positive for $\mu \neq (LSL + USL)/2$. When $\mu = (LSL + USL)/2$, the bias is positive for $n = 10$, but becomes negative for larger numbers. The bias goes to zero, however, as n becomes large. The variance of $\hat{C}pk$ increases as d/σ increases, but decreases as $\left| \mu - \frac{1}{2}(LSL + USL) \right|$ increases. It also decreases as n increases. Even when n is as large as

40, there is substantial uncertainty in $\hat{C}pk$ and Kotz and Johnson deem it unwise to use it as a preemptory guide to action.

Chou and Owen (1989) derive the distribution of $\hat{C}pk$ of equation (2.9) through the joint distribution of $(\hat{C}pl, \hat{C}pu)$, where $\hat{C}pk = \min\{\hat{C}pl, \hat{C}pu\}$. This joint distribution is complicated since $(\hat{C}pl, \hat{C}pu)$ are two dependent noncentral t variables. Guirguis and Rodriguez (1992) use the formula of Chou and Owen as the basis for a computer program. They give graphs of the probability density function of $\hat{C}pk$ for selected parameter values and sample sizes of 30 and 100.

Bissell (1990) uses the modified estimator

$$\hat{C}'pk = \begin{cases} \dfrac{USL - \overline{X}}{3\hat{\sigma}} & \text{for} \quad \mu \geq \dfrac{LSL + USL}{2} \\ \dfrac{\overline{X} - LSL}{3\hat{\sigma}} & \text{for} \quad \mu \leq \dfrac{LSL + USL}{2} \end{cases}. \tag{2.10}$$

This differs from $\hat{C}pk$ only in the use of μ in the place of $\overline{X}$. Kotz and Johnson (1993) point out that $\hat{C}'pk$ cannot be calculated unless μ is known, in which case one would not need to estimate it. The distribution of $\hat{C}'pk$ in equation (2.10) is proportional to a noncentral t with $(n-1)$ degrees of freedom and noncentrality parameter $\dfrac{\sqrt{n}}{\sigma}\left\{d - \left|\mu - \dfrac{LSL + USL}{2}\right|\right\}$. Although $\hat{C}'pk$ is not of practical use, it will not differ greatly from $\hat{C}pk$, except when $\mu \approx (LSL + USL)/2$. Unfortunately, a process mean μ close to the specification interval midpoint m is precisely the situation hoped for by a process manager.

Zhang *et al.* (1990) and Kushler and Hurley (1992) give confidence intervals for Cpk which are complicated to compute since they involve a noncentral t, but Kushler and Hurley suggest as an approximation

$$\hat{C}pk\left\{1 \pm z_{1-\alpha/2}\left\{\frac{n-1}{n-2} - \frac{n-1}{2}\left\{\frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\right\}^2\right\}^{1/2}\right\}. \tag{2.11}$$

Heavlin (1988) suggests

$$\hat{C}pk \pm z_{1-\alpha/2}\left\{\frac{n-1}{9n(n-3)} + \hat{C}^2pk\frac{(n-5)}{2(n-1)(n-3)}\right\}^{1/2} \tag{2.12}$$

as an approximate 100(1 - α)% confidence interval for *Cpk*. Chou *et al.* (1990) provide tables of approximate 95% lower confidence limits for *Cpk* under the assumption of normality.

Franklin and Wasserman (1992) carried out a simulation study to assess the properties of these limits and discovered that they are conservative. They found the actual coverage for the limits to be about 96%. They provide the lower bound

$$\hat{C}pk - z_{1-\alpha}\left\{\frac{1}{9n} + \frac{\hat{C}^2pk}{2(n-1)}\right\}^{1/2}, \tag{2.13}$$

which they found to produce accurate results for n at least 30. Kushler and Hurley (1991) suggest the simpler formula

$$\hat{C}pk\left\{\frac{1-z_{1-\alpha}}{\sqrt{2(n-1)}}\right\}, \tag{2.14}$$

while Kotz and Johnson (1993) report good results for the two-sided confidence interval

$$\hat{C}pk\left\{\frac{1 \pm z_{1-\alpha/2}}{\sqrt{2(n-1)}}\right\}, \tag{2.15}$$

which first appeared in Heavlin (1988).

In a recent article, Pearn and Chen (1997) present a case study using hypothesis testing to determine the capability of a process which produces rubber strip components for audio speakers. The measured characteristic is the weight of the rubber strip and for each speaker model, a unique production specification [LSL, USL] is set. The authors discuss three common estimators of Cpk. The first, proposed by Bissell (1990), is given by

$$\hat{C}'pk = \begin{Bmatrix} \dfrac{USL - \overline{X}}{3\hat{\sigma}} & \text{for} \quad \mu \geq \dfrac{LSL + USL}{2} \\ \dfrac{\overline{X} - LSL}{3\hat{\sigma}} & \text{for} \quad \mu \leq \dfrac{LSL + USL}{2} \end{Bmatrix}. \tag{2.16}$$

The second estimator is the natural estimator given by Kotz and Johnson (1993),

$$\hat{C}pk = \frac{d - \left| \overline{X} - \frac{1}{2}(LSL + USL) \right|}{3\hat{\sigma}}. \tag{2.17}$$

The third estimator, newly proposed by Pearn and Chen (1997), is given by

$$\hat{C}''pk = \frac{\left\{ d - (\overline{X} - m) I_A(\mu) \right\}}{3S}, \tag{2.18}$$

where

$$I_A(\mu) = \begin{Bmatrix} 1 & \text{if } \mu \in A = \{\mu \mid \mu \geq m\} \\ -1 & \text{if } \mu \notin A \end{Bmatrix}. \tag{2.19}$$

Pearn and Chen (1997) report that Bissell's estimator $\hat{C}'pk$ assumes the knowledge of whether the process mean is above or below the specification midpoint, whereas their own Bayesian-like estimator $\hat{C}''pk$ only requires the knowledge of $\Pr[\mu \geq m]$ or $\Pr[\mu < m]$. This presumably may be obtained from historical information. Kotz and Johnson (1993) had previously investigated their natural estimator and were able to show that, while both their estimator and Bissell's are biased, the variance of their $\hat{C}pk$ is smaller than Bissell's $\hat{C}'pk$.

Pearn and Chen (1997) show that under the assumption of normality, the distribution of the estimator $3(n)^{1/2}\hat{C}''pk$ is $t_{n-1}(\delta)$, a noncentral t with (n - 1) degrees of freedom and noncentrality parameter $\delta = 3(n)^{1/2}Cpk$. Pearn and Chen (1997) also show that $\tilde{C}pk = b_f \hat{C}''pk$ becomes an unbiased estimator of Cpk, where the "debiasing factor" b_f is given by

$$b_f = \sqrt{\frac{2}{n-1}}\frac{\Gamma[(n-1)/2]}{\Gamma[(n-2)/2]}. \tag{2.20}$$

Furthermore, the variance of $\tilde{C}pk$ is smaller than that of the estimators of Bissell (1990) or Kotz and Johnson (1993). The authors opt for their own $\tilde{C}pk$ in their case study application.

Pearn and Chen (1997) point out that looking at a single point estimate of Cpk from sample data and making a decision is highly unreliable. They present a procedure for engineers to correctly determine whether their processes meet the capability requirement preset in the factory. To determine whether a given process is capable, they consider the statistical hypothesis H_0: $Cpk \le C$ (process is not capable). Based on the sampling distribution of $\hat{C}''pk$, the rejection probability, commonly called the p value, can be evaluated as

$$p \text{ value} = \Pr\left[\tilde{C}pk \ge W \middle| Cp \le C\right] = \Pr\left[t_{n-1}(\delta) \ge 3n^{1/2}Wb_f^{-1} \middle| Cpk \le C\right], \tag{2.21}$$

where the observed value of $\tilde{C}pk$ is W. The authors give extensive tables of p values for various sample sizes, a range of values W, and C equal to 1.00 or 1.33. Their four step procedure is (1) decide the definition of "capable," setting C to 1.00 or 1.33 and α-risk to 0.01 or 0.05, (2) calculate the value W from the sample, (3) check the table based on the value W and sample size n, and (4) conclude the process is capable if the p value from the table is less than α.

2.3. The *Cpm* Index and Its Variants

The Cpm index was introduced by Chan *et al.* (1988a) in the literature but was first proposed by Hsiang and Taguchi (1985) at the ASA, Las Vegas. While Hsiang and Taguchi were motivated exclusively by loss function considerations, Chan was interested in comparisons with Cpk. The original definition has

$$Cpm = \frac{USL - LSL}{6\sqrt{\sigma^2 + (\mu - T)^2}} = \frac{d}{3\sqrt{\sigma^2 + (\mu - T)^2}}, \tag{2.22}$$

where T is typically $(LSL+USL)/2$. If it is not, it can be a very poor measure. Chan *et al.* (1988b) give a modified version,

$$C^*pm = \frac{\min\{USL - T, T - LSL\}}{3\sqrt{\sigma^2 + (\mu - T)^2}} = \frac{d - |T - m|}{3\sqrt{\sigma^2 + (\mu - T)^2}}. \tag{2.23}$$

If $T = m$, then $C^*pm = Cpm$. If not, then it has the same potentially poor performance as Cpm. Since $\sigma^2 + (\mu - T)^2 \geq \sigma^2$, we have $Cpm \leq Cp$. If $T = \mu$, then Cpm is the same as Cp. If, also, $\mu = m$, then both Cpm and C^*pm are the same as Cpk. If, further, the process characteristic has a stable normal distribution, then a value of 1.00 for all four indices means that the proportion of product within specification limits is about 99.73%. Negative values are impossible for Cp and Cpm. They are impossible for C^*pm if T falls within specification limits. They are impossible for Cpk if μ falls within specification limits.

Kotz and Johnson (1993) point out that if the loss function is proportional to $(X - T)^2$, then the denominator of C^*pm is a measure of average loss, and so there is no need for *USL*, *LSL*, or the factor 6. They question the use of a cumbersome Cpm or C^*pm rather than the loss function itself. In the final analysis, as Kushler and Hurley (1992) state, the main distinction between the Cpm and Cpk is that the Cpk emphasizes the $(USL - LSL)$, while the Cpm uses T to scale the loss function in the denominator. If T is not equal to m, "... moving the process mean towards the target (which will increase the Cpk and Cpm indices) can reduce the fraction of the distribution within specification limits." Kotz and Johnson (1993) provide point estimators for Cpm and C^*pm (when X is normal) which are functions of noncentral chi-squareds, each of which is, in turn, an infinite Poisson-weighted average of central chi-squareds. Confidence interval estimation is, of course, no more tractable.

Boyles (1992) proposed the index

$$C^{+}pm = \frac{1}{3}\left[\frac{E_{X<T}\left[(X-T)^2\right]}{(T-LSL)^2} + \frac{E_{X>T}\left[(X-T)^2\right]}{(USL-T)^2}\right]^{-1/2} \tag{2.24}$$

for use when $\mu \neq T$. The expectation

$$E_{X<T}\left[(X-T)^2\right] = E_{X<T}\left[(X-T)^2 | X < T\right]\Pr[X<T] \tag{2.25}$$

is the semivariance, introduced for an asymmetric loss function. He gives a natural estimator

$$\hat{C}^{+}pm = \frac{1}{3}\left[\frac{\sum_{X_i<T}(X_i-T)^2}{n(T-LSL)^2} + \frac{\sum_{X_i>T}(X_i-T)^2}{n(USL-T)^2}\right]^{-1/2} \tag{2.26}$$

Even for a normal process characteristic, its distribution is complicated. Kotz and Johnson (1993) derive expressions for the mean and variance under the very restrictive conditions that $\mu = m = T$.

Pearn *et al.* (1992) give the index

$$Cpmk = \frac{d - |\mu - m|}{3\sqrt{E(X-T)^2}} = \frac{d - |\mu - m|}{3\sqrt{\sigma^2 + (\mu - T)^2}}. \tag{2.27}$$

They note that $Cpmk \leq Cpm$ and also $Cpmk \leq Cpk$. Whenever $\mu = m = T$, then $Cp = Cpk = Cpm = Cpmk$. The estimator of $Cpmk$ has a complex distribution. Its expected value is a product of betas and chi-squareds.

Kotz and Johnson (1993) suggest a class of PCIs defined by $Cpm(a) = \left(1 - a\zeta^2\right)Cp$, where $|\zeta| = \frac{1}{\sigma}|\xi - T|$ is small. The positive constant a is chosen to balance variability and departure from target. A natural estimator is given by

$$\hat{C}pm(a) = \left\{1 - a\left\{\frac{\overline{X} - T}{S}\right\}^2\right\}\hat{C}p. \tag{2.28}$$

This estimator is distributed as a complex function of chi-squareds and normals. The *Cpm*(a) has the same drawbacks as the *Cpm*. If T is not equal to m, $T - \delta$ and $T + \delta$ can correspond to different proportions NC items.

In a recent paper, Zimmer and Hubele (1997) provide tables for computing quantiles of the sampling distribution of the sample estimator of *Cpm*. The application of these values to hypothesis testing is illustrated.

It is difficult for us to work up any enthusiasm for the *Cpm* index or any of its progeny. The *Cpm* index confounds, into one summary measure, the two subproblems of estimating the proportion π_0 of product outside specification and estimating monetary or related loss. Now clearly each of these subproblems is important. Yet we feel that the *Cpm* index and its variants are too difficult to estimate and understand. It would seem that the better course would be to directly attack estimation of the long-run behavior of the random characteristic X through estimates of (μ, σ). From these estimates, an estimate of the proportion π_0 of product outside specification can be found. The analysis of loss can then be addressed.

It is not always clear from the literature exactly whose loss is to be measured. Is it the supplier's loss or is it the customer's loss? If the *Cpm* index is meant to communicate information between a supplier and outside customers, should not the customer's loss be the important focal point? The literature of *Cpm* seems to concentrate on the supplier's loss. Suppose the supplier computes a *Cpm* tailored to one of his customers. But now he must construct a second *Cpm* for a second customer who has a different loss function, *et cetera*.

Perhaps it is best for the supplier to concentrate on estimating π_0, a concept which should migrate across customers more easily. If the customer is truly indifferent to any X inside the specification interval, then the case can be made that his relevant measure is the supplier's proportion π_0 of product outside specification. Together with a rate of production and data such as selling price, materials costs, labor costs, overhead costs, costs of rework for product outside specification, and the like, he can, in theory, determine cash inflows and outflows attributable to the product. Admittedly, these figures may be less than sharp.

2.4. The Effect of Autocorrelation on Process Capability Indices

Yang and Hancock (1990) have shown that for an autocorrelated sample $\{X_i\}_n$ identically distributed from a population with mean μ and variance σ^2, the usual sample variance $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$ has expected value $ES^2 = (1-\overline{\rho})\sigma^2$, where

$$\overline{\rho} = \frac{1}{n(n-1)} \sum_{i \neq j}^{n(n-1)} \rho_{ij} = \frac{2}{n(n-1)} \sum_{i<j}^{n(n-1)/2} \rho_{ij} \qquad (2.29)$$

is the average of the $n(n - 1)$ pairwise correlation parameters of the model. For positively correlated $\{X_i\}_n$, this implies that S will tend to underestimate σ and so $\hat{C}p = (USL - LSL)/6S$ will tend to overestimate Cp. This reinforces the bias in $1/S$, which tends to overestimate Cp. The effect is similar for the Cpk index.

In a recent article, Shore (1997) uses Monte Carlo simulation to demonstrate how autocorrelations may adversely affect the sampling distributions of Cp and Cpk, and suggests remedies. He uses the autocorrelation function $\rho(k) = \exp(-\lambda k)$ for various positive λ and n to simulate autocorrelated data. He concludes that (1) the capability indices are biased upward, the bias diminishing with increasing n, and (2) the standard errors of the indices increase considerably when autocorrelation is present, relative to an uncorrelated process, particularly for n less than fifteen. He distinguishes between instantaneous process capability and long-term capability, and suggests two remedies. A model-dependent remedy approximates the autocorrelation population parameter and adjusts the estimated process variance accordingly. A model-free remedy seeks to adjust the estimate of the population variance by rational subgrouping of samples from the process.

2.5. Non-Normality and Robustness

Marcucci and Beazley (1988) have devised PCIs for attributes. They consider

$$R = \frac{\omega(1-\omega_1)}{(1-\omega)\omega_1} = \frac{\omega/(1-\omega)}{\omega_1/(1-\omega_1)}, \tag{2.30}$$

where ω is the actual NC and ω_1 is the maximally acceptable NC. An R of 2 is indicative of a poor process. An estimator of R is given by

$$\hat{R} = \frac{X+\frac{1}{2}}{n-X+\frac{1}{2}}\left\{\frac{1-\omega_1}{\omega_1}\right\}. \tag{2.31}$$

Unfortunately, the variance of this estimator is large and so is its positive bias.

In his seminal paper, Kane (1986) devoted only a short paragraph to the effects of non-normality of the measured characteristic X on properties of capability indices. He suggests using the proportion of product outside specification as a capability index.

Gunter (1989) has studied the interpretation of Cpk under three different non-normal distributions, (1) a skewed distribution (chi-squared), (2) a heavy-tailed distribution (t), and (3) a uniform distribution. These three were standardized to have a mean of zero and a standard deviation of one. The NCPPM for plus or minus 3σ are 14000 for (1), 4000 for (2), and 0 for

(3). Recall that a standard normal has 2700 NCPPM by comparison. He uses the word "hopelessness" in describing any attempt at meaningful interpretation.

English and Taylor (1990) have carried out extensive Monte Carlo studies of the distribution of $\hat{C}p$ for normal, symmetrical triangular, uniform, and exponential distributions, using sample sizes of n = 5, 10, 30, and 50. For n less than 30, there can be substantial departures from the true Cp value. In addition, the values for the exponential distributions differ sharply from the values for the three symmetrical distributions.

Kocherlakota *et al.* (1992) have established the distribution of $\hat{C}p = d/3\hat{\sigma}$ in two cases, when the process distribution is (i) contaminated normal with $\sigma_1 = \sigma_2 = \sigma$ and (ii) contaminated with an Edgeworth series

$$f_X(x) = \left(1 - \tfrac{1}{6}\lambda_3 D^3 + \tfrac{1}{24}\lambda_4 D^4 + \tfrac{1}{72}\lambda_3^2 D^6\right)\phi[x; 0, 1]. \tag{2.32}$$

Here D^j is the jth derivative with respect to X and λ_3 and λ_4 are standardized measures of skewness and kurtosis. Kotz and Johnson (1993) derive moments of $\hat{C}p$ for a more general contamination model with k components (but with each component having the same variance). The $E\left[\hat{C}_p^r\right]$ is a multinomially-weighted average of noncentral chi-squareds. Their main observation is that this contaminated version of $\hat{C}p$ can have a negative bias whereas the uncontaminated $\hat{C}p$ always has a positive bias. Kocherlakota *et al.* (1992) derive the moments of $\hat{C}pu = \left(USL - \overline{X}\right)/3\hat{\sigma}$. The distribution of this estimator is a mixture of doubly noncentral t distributions.

Price and Price (1992) estimate by simulation, the expected values of $\hat{C}p$ and $\hat{C}pk$ for several process distributions including normals, uniforms, betas, and gammas. They used $E\left[\hat{C}p/Cp\right] = E\left[\hat{C}p\right]/Cp$ and $E\left[\hat{C}pk/Cpk\right] = E\left[\hat{C}pk\right]/Cpk$ to compare distributions. On comparing the estimates for the normal distribution with the correct values, they found that the estimates are in excess by about 2% for n = 10 and 0.5% for n = 30 or 100. Sampling variation was also evident in the nine gammas they examined. As skewness increases, so does $E\left[\hat{C}p/Cp\right]$, reaching a remarkably high bias even for n as large as 100. The same holds for $E\left[\hat{C}pk/Cpk\right]$.

Clements (1989) proposed a method of construction based on the assumption that the process distribution can adequately be represented by a Pearson distribution. The aim is to replace the 6 in the denominator of Cp by numbers θ_l and θ_u such that

$$\Pr\left[\xi - \theta_l\sigma \leq X \leq \xi + \theta_u\sigma\right] = 0.0027. \tag{2.33}$$

In calculating Cpk, Clements suggests using the same value of θ, but replacing the sample mean with the sample median. Johnson *et al.* (1992) suggest using

$$Cp(\theta) = \frac{USL - LSL}{\theta\sigma} = \frac{2d}{\theta\sigma}, \tag{2.34}$$

where θ is now chosen so that the "capability" is not greatly affected by the shape of the process distribution. Reflecting on the two methods of Clements and Johnson, we see that Clements makes a direct allowance for the values of the skewness and kurtosis coefficients, while Johnson aims at getting limits which are insensitive to these values. But each method relies on the assumption that the population distribution has a unimodal shape close to a Pearson (Clements) or a gamma (Johnson).

Chan *et al.* (1988) proposed using distribution-free tolerance intervals to obtain "distribution-free" PCIs. These tolerance intervals are designed to include at least β of a distribution with preassigned probability $(1 - \alpha)$ for a given β close to one and α close to zero. A natural choice for β is 0.9973. Unfortunately, construction of such intervals require prohibitively large samples of size 1000 or more. Chan *et al.* (1988) suggest an alternative that would require smaller sample sizes of around 300. They recommend taking $\beta = 0.9546$ and using 1.5 times the tolerance interval length, or $\beta = 0.6826$ and using 3 times the tolerance interval length, in place of $6\hat{\sigma}$. The reasoning behind these heuristics comes, of course, from the normal distribution, which would no longer make them "distribution free."

Franklin and Wasserman (1991), together with Price and Price (1992), are the pioneers in the application of bootstrap methods to the estimation of the *Cpk*. However, experience has indicated that a minimum of one thousand bootstrap samples are needed for a reliable calculation of bootstrap confidence intervals for the *Cpk*. The difficulties in this approach come about because the indices are ratios of random variables with a large amount of variability in the denominator. Similar difficulties arise in estimating a correlation coefficient or the ratio of two expected values. Cochran (1963) writes of the distribution of ratio estimators, "the known theoretical results fall short of what we would like to know for practical applications."

Rodriguez (1992) reports various attempts to extend the definitions of standard capability indices to non-normal distributions. The idea is to generalize the formulas for the standard indices by replacing 3σ with percentiles, as in

$$Cpk = \min\left\{\frac{USL - P_{0.5}}{P_{0.9987} - P_{0.5}}, \frac{P_{0.5} - LSL}{P_{0.5} - P_{0.0013}}\right\}. \tag{2.35}$$

This extension applies to unimodal skewed distributions. It simplifies to the familiar *Cpk* when the distribution is normal. As noted by Pearn *et al.* (1992), the basic disadvantage of percentile-based extensions is that the extreme percentiles $P_{0.0013}$ and $P_{0.9987}$ cannot be estimated precisely unless a massive number of process measurements are available. This problem can be circumvented if one assumes a parametric distribution for the data and computes the percentiles from the fitted distribution. Pearn *et al.* (1992) believe that by working with distributions from a large class such as the Pearson system or Johnson system, one can fit data distributions with a wide variety of shapes. Rodriguez (1992) answers that equations for fitted Pearson or Johnson curves are "generally complicated and difficult to interpret." He believes that simpler distributions, chosen from a smaller family of distributions such as the gamma, lognormal, or Weibull, can serve as useful models for the process distribution, while providing reasonable curve flexibility. He notes that the gamma family is the Pearson Type III family, and the lognormal family is the Johnson S_L family. The Weibull family, however, does not belong to either system.

In a recent article, Sarker and Pal (1997) consider measuring the concentricity of a circular machined component. Concentricity generally does not follow the normal distribution pattern, but is explained through an extreme value distribution of either Type 1,

$$F(x) = \exp\left\{-\exp\left\{-\frac{x-\alpha}{\theta}\right\}\right\}, \quad 0 \le x < \infty \tag{2.36}$$

or Type 2,

$$F(x) = \exp\left\{-\left\{-\frac{x-\alpha}{\theta}\right\}^{-k}\right\}, \quad \alpha \le x < \infty. \tag{2.37}$$

The authors discuss three proposed methods commonly used for non-normal distributions. In Method 1, the capability index taken is

$$CPU = \frac{USL - \overline{X}}{4S}, \tag{2.38}$$

where $\overline{X}$ is the sample mean, S is the commonly used sample standard deviation, and $4S$ is chosen instead of $3S$ in the denominator because of skewness. The process is considered capable if CPU is at least one. In Method 2 *per* Owen (1962), a non-normal curve is fitted to the data, and the proportion of product outside specification is estimated. That proportion nonconforming is converted to an equivalent z-score for a normal distribution. The z-score is then divided by an appropriate factor to get the value of the process capability index CPU. The process is considered capable if CPU is at least one. Method 3, which corresponds to the suggestion of Clements (1989), uses the Pearson family of curves to approximate almost any distribution. From the sample data, the mean, standard deviation, skewness, and kurtosis are calculated. Based on the skewness and kurtosis values, standardized 99.865 and 50 percentile values are read from a table to calculate the estimated 99.865 and 50 percentiles. The process capability index CPU is calculated as

$$CPU = \frac{USL - M}{U_p - M}, \tag{2.39}$$

where U_p is the estimated 99.865 percentile and M is the estimated 50 percentile. The process is considered capable if CPU is at least one.

Although Method 1 is used extensively by many industries because of its simplicity, the authors point out that there is no known statistical basis for the denominator. Methods 2 and 3 are too laborious to do manually. This leads the authors to suggest a fourth method for measuring capability. An extreme value Type 1 distribution, whose parameters (α, θ) are estimated from the sample, is fitted to the data. The index is computed as

$$CPU = \frac{USL - x_{0.5}}{x_{0.99865} - x_{0.5}}, \tag{2.40}$$

where x_i is the ith percentile.

CPU indices are estimated under the four methods for three data sets of varying skewness, kurtosis, and percentage nonconforming product in the sample. The authors observe that Method 2 clearly underestimates the *CPU*, whereas Method 3 overestimates the *CPU*. The authors use simulation to give 95 and 99 percent critical lower bounds for the true *CPU* on their Method 4 for various sample sizes.

In a recent article, Somerville and Montgomery (1996) examined four non-normal distributions for their effect on inferences made using the standard capability indices, *Cp* and *Cpk*. The gamma, lognormal, Weibull, and *t* distributions were analyzed. Various shapes for each of the four distrbutions were considered. The errors associated with a normal assumption on a non-normal distribution were evaluated and tabulated. The proportions nonconforming, in parts per million, were compared to those obtained with the normal distribution. The errors were found to be extremely large in most instances. The skewness and kurtosis of the non-normal distributions contribute to a significant difference in the defect percentage. The authors recommend that a sample distribution for which a capability estimate is desired should be evaluated for departures from normality. The authors also recommend that methods which compensate for non-normality should be considered if a high degree of confidence is to be placed on the capability estimates.

2.6. Multivariate Process Capability Indices

Frequently, a manufactured item possesses several measurable quality characteristics. These measurements must be separately and simultaneously controlled since the assessed quality of the ultimate product is a function of these combined effects. Only recently have multivariate PCIs been investigated. It would seem obvious that the drawbacks observed for the univariate PCIs can only multiply. As it turns out, these "multivariate" indices are often simply univariate indices in disguise.

If $\{X_i\}_\nu$ are the measured variates, the usual practice is to look at the ν-dimensional rectangular parallelepiped defined by

$$\{LSL_i \le X_i \le USL_i\}_\nu . \tag{2.41}$$

Chan *et al.* (1990) suggest using the product of the ν univariate *Cpm* values as a multivariate PCI. But this will not do, even if the variates are independent. A very small *Cpm* in one variate can be compensated by a very large *Cpm* in another variate. This is clearly an unsatisfactory quality for an index to have.

Kotz and Johnson (1993) stress that any single index of process capability based on multivariate characteristics has an even higher risk of misuse and misinterpretation than is the case for univariate PCIs. But just one year earlier with Pearn *et al.* (1992), they formulated just such an index. As in Chan *et al.* (1990), they begin with the assumption that ν characteristics $(X_1, X_2, \ldots, X_\nu)$ are measured for each of n items and that the specification limits for these characteristics are prescribed in the form $(\mathbf{X}-\mathbf{T})^t\,\mathbf{A}^{-1}(\mathbf{X}-\mathbf{T}) \le c^2$ for vectors $\mathbf{X}^t = (X_1, \ldots, X_\nu)$, $\mathbf{T}^t = (T_1, \ldots, T_\nu)$, and the $\nu \times \nu$ positive definite matrix $\mathbf{A}$. The specification region is an ellipse in the (X_1, X_2) plane for $\nu = 2$ and a ν-dimensional ellipsoid for $\nu > 2$.

In a recent article, Byun *et al.* (1997) develop a probability vector which assesses the producibility of a product design with given specifications in terms of process capability and manufacturing cost. Quality characteristics are classified into three categories, critical characteristics $\{X_i\}_{n_1}$, major characteristics $\{Y_i\}_{n_2}$, and minor characteristics $\{Z_i\}_{n_3}$, according to their relative importance. A PCI for each of the $(n_1+n_2+n_3)$ characteristics is computed. Then a PCI for each of the three categories is calculated by

$$\left(PCI_{critical}, PCI_{major}, PCI_{minor}\right)$$

$$=\left(\min\left\{PCI_{X_i}\right\}_{n_1}, \min\left\{PCI_{Y_i}\right\}_{n_2}, \min\left\{PCI_{Z_i}\right\}_{n_3}\right). \tag{2.42}$$

A weighted geometric mean of all $(n_1 + n_2 + n_3)$ PCIs is calculated as

$$PCI_{overall} = \left[\left(PCI_{X_1} \cdots PCI_{X_{n_1}}\right)^{1/n_1}\right]^{w_1/\sum w_i} \left[\left(PCI_{Y_1} \cdots PCI_{Y_{n_2}}\right)^{1/n_2}\right]^{w_2/\sum w_i}$$

$$\times\left[\left(PCI_{Z_1} \cdots PCI_{Z_{n_3}}\right)^{1/n_3}\right]^{w_3/\sum w_i}. \tag{2.43}$$

Manufacturing cost is also considered and is included as MC, the fifth element of a probability vector. The probability vector is given by

$$PV = \left[PCI_{critical}, PCI_{major}, PCI_{minor}, PCI_{overall}, MC\right]. \tag{2.44}$$

The authors claim that the vector can be used for determining a good design which has high process capability and low manufacturing cost. The authors demonstrate its application to the production of beef stew and ham slice pouches.

2.7. Recent Approaches and Criticism

In the first of two articles, Levinson (1997a) gives exact confidence limits, in the case of *iid* normal observations, for the index $Cp = (USL - LSL)/6\sigma$ and also exact confidence limits for $Cpl = (\mu - LSL)/3\sigma$ and $Cpu = (USL - \mu)/3\sigma$. Of course, $Cpk = \min\{Cpl, Cpu\}$. The interval for Cp is simple enough. The intervals for Cpl and Cpu are more difficult, involving the noncentral t. In addition, Levinson gives one-sided tolerance factors.

In the companion article, Levinson (1997b) compares two approximate lower confidence limits for Cpk under *iid* normal sampling. Kushler and Hurley (1992) *per* Zhang *et al.* (1990) give

$$c_k = \hat{C}pk\left\{1 - z_{1-\alpha}\left\{\frac{n-1}{n-2} - \frac{n-1}{2}\left\{\frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\right\}^2\right\}^{1/2}\right\} \tag{2.45}$$

as the lower $100(1-\alpha)\%$ confidence limit for *Cpk*. On the other hand, Franklin and Wasserman (1992) *per* Bissell (1990) recommend

$$c_k = \hat{C}pk - z_{1-\alpha}\left\{\frac{1}{9n}+\frac{\hat{C}_{pk}^2}{2n-2}\right\}^{1/2} \quad (2.46)$$

Comparing the two approximations to the exact noncentral *t* result for $\hat{C}pk = 1$, Levinson draws several conclusions. First, each approximation improves with an increase in sample size. Second, even with $n = 200$, the approximations provide only 95 percent assurance that the true *Cpk* is better than 0.9 when $\hat{C}pk = 1$. Third, point estimates of *Cpk* based on *n* of 10 or 20 are meaningless. Fourth, the equation (2.45) approximation underestimates, while equation (2.46) overestimates, the lower $100(1-\alpha)\%$ confidence limit for *Cpk* as computed from the true noncentral *t*.

Tsui (1997) *per* Kane (1986) summarizes the applications of process capability indices as (1) comparing process performance, (2) comparing among different processes, (3) providing information about proportion conforming or closeness to target, and (4) providing directions for quality improvement. He separates "proportion conforming" approaches from "loss function" approaches. He believes in measuring loss more directly, rather than incorporating loss into a PCI, and proposes "yield" as the most natural index for measuring conforming product. He defines yield Y by

$$Y = \int_{LSL}^{USL} dF(x), \quad (2.47)$$

where $F(x)$ is the cumulative distribution function of the process characteristic X. Tsui also defines the complementary measure, potential yield, given by

$$Y_p = \int_{\mu-d}^{\mu+d} dF(x), \tag{2.48}$$

where $d = (USL - LSL)/2$ is the half-width of the specification interval. Tsui claims that together (Y_p, Y) provide the same information about the process as (Cp, Cpk), but does not require a normality assumption. Of course, it does require symmetry in an assumed known $F(x)$.

Kaminsky, Dovich, and Burke (1998) give an overview of process capability indices currently in use and discuss problems with them. They point out that most of the current indices are applicable only for normally distributed processes in a state of statistical control. They show that Cpk is an ambiguous measure in the sense that two processes with the same Cpk can have a different number of nonconforming parts per million items that leave the plant. They also show, by example, that the alternative methods of estimating the true Cpk of the process can produce different estimates of Cpk. In addition, the estimators for Cpk exhibit considerable variability, and a single point estimate of the process capability can be significantly different from the true value. They describe a procedure to conduct a process capability analysis, without the use of any process capability index, using the binomial distribution and exceedance probabilities to describe the number of nonconforming items that leave the plant. For example, if the measured characteristic X is Weibull with probability density function given by

$$f(x) = k(x/\theta)^{k-1}\theta^{-1}\exp\left\{-(x/\theta)^k\right\}, \quad x \geq 0, \tag{2.49}$$

then the proportion p of nonconforming product is

$$p = 1 - \int_{LSL}^{USL} f(x)dx. \tag{2.50}$$

If it is common to send out shipments with a lot size of N items and if T is the total number of nonconforming items in a lot of N items, then

$$\Pr[T = t] = \binom{N}{t} p^t (1-p)^{N-t}, \quad t = 0, 1, 2, \ldots, N \tag{2.51}$$

and

$$\Pr[T > t] = \sum_{i=t+1}^{N} \binom{N}{i} p^i (1-p)^{N-i}, \quad t = 0, 1, 2, \ldots, N. \tag{2.52}$$

With this procedure, according to Kaminsky *et al.* (1998), the use of any process capability index is "totally unnecessary, and a strong argument is provided to eliminate the use of process capability indices."

CHAPTER 3. CAPABILITY INDICES UNDER NORMAL INDEPENDENCE

We begin by examining the assumption of the normality of the measured characteristic X. There are often justifiable reasons for this model. When normality is met in practice, a well-developed body of knowledge is available to draw upon. Of course, practitioners must always be aware of the possible consequences from using the wrong model.

3.1. Why Some Real World Data Exhibit Normality

This is not the place for a rigorous discussion of the limit theorems of probability and statistics. Their statement and full import are neither elementary nor concise. We refer the reader to Billingsley (1986), Lehmann (1983), Rohatgi (1976), or Serfling (1980). Nevertheless, a very small background is needed in order to proceed.

Random variables are functions (from the sample space of outcomes into a field of numbers) and there are several ways in which the limit of a sequence of functions may be defined. The common types of convergence are (1) almost sure convergence, (2) convergence in mean square, (3) convergence in probability, and (4) convergence in distribution. Let us point out that type (1) implies (3), (2) implies (3), (3) implies (4), and when the limit function is constant, (4) implies (3). Our concern is with (4) convergence in distribution.

The sequence of random variables $\{Y_n\}$ is said to converge to Y in distribution, if and only if at each point λ where Y is continuous,

$$\lim_{n\to\infty} F_{Y_n}(\lambda) = F_Y(\lambda). \tag{3.1}$$

The important point here is that for large but finite n, the probability $F_{Y_n}(\lambda) = \Pr[Y_n \le \lambda]$ can be approximated by the probability $\Pr[Y \le \lambda] = F_Y(\lambda)$, which may be simpler.

When one talks of the *central limit theorem* (CLT), one is really invoking a group of results whose commonality is *convergence in distribution* to a *normal* random variable. The Lindeberg-Levy version of the CLT states that for $\{X_n\}$ independent and identically distributed with mean μ and finite standard deviation σ, the normalized sequence of random variables

$$\{Z_n\} = \left\{ \frac{1}{\sigma\sqrt{n}} \left[\sum_{i=1}^{n} X_i - n\mu \right] \right\} = \left\{ \frac{\sqrt{n}}{\sigma} \left[\frac{1}{n} \sum_{i=1}^{n} X_i - \mu \right] \right\} \tag{3.2}$$

converges in distribution to the standard normal random variable.

This has been generalized in several directions. For example, it is not necessary that the sequence of random variables be identically distributed nor even independent. In fact, the finite variance requirement can be dropped. Each generalization requires additional requirements on higher moments. See Serfling (1980) for a complete survey.

More generally, a sequence of random variables $\{X_n\}$ is asymptotically normal with "mean" μ_n and "standard deviation" σ_n, if $\{(X_n - \mu_n)/\sigma_n\}$ converges in distribution to the standard normal random variable. Here $\{\mu_n\}$ and $\{\sigma_n\}$ are sequences of constants, not necessarily the mean and standard deviation of $\{X_n\}$.

Roughly speaking, when a random variable represents the total effect of a large number of small causes, none of which dominates, the central limit theorem asserts that the distribution of that variable tends to the normal distribution. Suppose the random variable X_n can be written as $X_n = a_1Y_1 + a_2Y_2 + \cdots + a_nY_n$, where each of Y have finite variance. We form the sequence of partial sums

$$S_n = \frac{X_n - EX_n}{} = \frac{(a_1Y_1 + a_2Y_2 + \cdots + a_nY_n) - (a_1EY_1 + a_2EY_2 + \cdots + a_nEY_n)}{}$$

$$S_n = \frac{X_n - EX_n}{\sqrt{VarX_n}} = \frac{(a_1Y_1 + a_2Y_2 + \cdots + a_nY_n) - (a_1EY_1 + a_2EY_2 + \cdots + a_nEY_n)}{\sqrt{Var(a_1Y_1 + a_2Y_2 + \cdots + a_nY_n)}}$$

$$= \frac{(a_1 Y_1 - a_1 EY_1) + \cdots + (a_n Y_n - a_n EY_n)}{\sqrt{Var(a_1 Y_1 + \cdots + a_n Y_n)}} \quad (3.3)$$

If the sequence S_n converges in distribution to a standard normal random variable, we say that the central limit holds. We expect convergence of S_n to standard normal when no individual $a_i Y_i$ variable "contributes more" than the others.

Kececioglu (1991) considers how the normal random variable may be generated in the world, both naturally and as a human construct in the context of reliability. Whenever a wear-out process is involved in the determination of the time to failure, or whenever many additive effects are involved, or whenever units exhibit an increasing failure rate, time to failure may follow a normal distribution. For some random variables a normal law may be an acceptable approximation in the center, but not in the tails. He gives examples of applications. Incandescent lamps have exhibited normally distributed times to failure. Shoes, clothing, furniture, simple electronic parts, mechanical parts, and simple parts with homogeneous deterioration properties have been observed to have normally distributed times to failure. The stress at failure of many structural materials has been found to follow the normal distribution.

But not all random variables can be regarded as the sum of many small effects and so there is no theoretical reason to expect a normal distribution to result. Non-normality can result when one or more substantial or extreme effects are predominant, as would be the case when values of the variable are concentrated near the lower or upper boundary of the distribution. An interesting point to be made is that even if S_n does not converge in distribution to a standard normal random variable, it will converge to *some* $\mathbf{L}^2$-random variable if the Y_i's feeding into it are each $\mathbf{L}^2$, that is, if each Y_i has finite positive variance (Kreyszig, 1978). This is because the random variables of finite positive variance constitute a Banach space with the appropriate norm (with the appropriate inner product, they also make a Hilbert space). It follows that *any* L^2-random variable can be regarded as the sum of many small effects. In fact, the unique expression of an L^2-random variable as a linear combination of basis L^2-random variables can be viewed as a compendium, if you will, of limit results. This should serve as a warning against any hasty invocation of a normal central limit theorem effect.

3.2. Estimating Cp when μ is Known and σ is Unknown

Let the random variable X be normal with finite mean μ and finite positive standard deviation σ. Let the specification limits be finite, fixed, and known with LSL strictly less than USL. The index Cp is given by

$$Cp = \frac{USL - LSL}{6\sigma}. \tag{3.4}$$

It is a measure of *actual* process capability whenever the process mean μ is equal to the midpoint of the specification interval, that is, whenever $\mu = m = (LSL + USL)/2$. Otherwise, it is a measure of *potential* process capability, conditional on the ability to center the process mean μ at the midpoint m of the specification interval without disturbing the process standard deviation σ. Note that Cp is positive since each of its numerator and denominator is assumed positive.

If both μ and σ are known, then Cp is known and does not have to be estimated. Also, if μ is unknown and σ is known, then Cp is known and does not have to be estimated. In this section, we take up the case of known μ and unknown σ. This case is likely artificial, yet it may provide insight into the more realistic case of both μ and σ being unknown. It is also the traditional approach to looking at an estimation problem. With μ known and σ unknown, we take the statistic $S=\sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2}$ as our point estimator of σ. Note carefully μ in the definition of S.

It will be sometimes convenient and often times instructive to study the random variable $\hat{C}p/Cp$ (no problem occurs in division since Cp is assumed positive). For example, a realization $\hat{C}p/Cp$ of 1.05 means that the estimated Cp is 5 percent larger than the true Cp, while a $\hat{C}p/Cp$ of 0.90 indicates that the estimated Cp is 10 percent smaller than the true Cp. For the case of known μ and unknown σ, we have

$$\frac{\hat{C}p}{Cp}=\frac{(USL-LSL)/6S}{(USL-LSL)/6\sigma}=\frac{\sigma}{S}, \tag{3.5}$$

where again,

$$S=\sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2}.$$

It is well known that for $\{X_i\}_n$ distributed *iid* $N(\mu,\sigma^2)$, the random variable $\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2$ is distributed as χ_n^2, a chi-squared random variable with n degrees of freedom. Therefore we have $\hat{C}p/Cp=\sigma/S$ distributed as $\sqrt{n}\,\chi_n^{-1}$, the square root of n times an "inverted chi" with n degrees of freedom. The random variable χ_n^{-1} is simply the reciprocal of the square root of a chi-squared random variable, that is, $\chi_n^{-1}=1\Big/\sqrt{\chi_n^2}$.

To derive the probability density function of the inverted chi, we start with the pdf of the chi-squared. Let X be distributed chi-squared with n degrees of freedom. The pdf of X is given by

$$f_X(x)=\frac{x^{(n-2)/2}e^{-x/2}}{\Gamma[n/2]2^{n/2}} \tag{3.6}$$

for nonnegative real x and positive real n. It can be shown that $E[X]=n$ and $Var[X]=2n$.

By the change of variable $y = x^{-1/2}$, we get the pdf

$$f_Y(y) = f_X\, x(y)\left|\frac{dx}{dy}\right| = f_X\left(y^{-2}\right)2y^{-3} = \frac{y^{-(n+1)}e^{-1/2y^2}}{\Gamma[n/2]2^{(n-2)/2}} \tag{3.7}$$

for nonnegative real y and positive real n. To get the expected value of the inverted chi random variable Y,

$$E[Y] = \int_0^\infty y f_Y(y)dy = \int_0^\infty \frac{y^{-n}e^{-1/2y^2}}{\Gamma[n/2]2^{(n-2)/2}}dy$$

$$= \frac{\Gamma[(n-1)/2]2^{(n-3)/2}}{\Gamma[n/2]2^{(n-2)/2}} \int_0^\infty \frac{y^{-n}e^{-1/2y^2}}{\Gamma[(n-1)/2]2^{(n-3)/2}}dy\,.$$

The integral is equal to one, and so

$$E[Y] = E\left[\chi_n^{-1}\right] = \frac{\Gamma[(n-1)/2]}{\Gamma[n/2]}\frac{1}{\sqrt{2}}, \tag{3.8}$$

which is finite for real n greater than one.

We next get $E\left[Y^2\right]$ as an intermediate step toward the determination of $Var[Y]$,

$$E\left[Y^2\right] = \int_0^\infty y^2 f_Y(y)dy = \int_0^\infty \frac{y^{-n+1}e^{-1/2y^2}}{\Gamma[n/2]2^{(n-2)/2}}dy$$

$$= \frac{\Gamma[(n-2)/2]2^{(n-4)/2}}{\Gamma[n/2]2^{(n-2)/2}} \int_0^\infty \frac{y^{-n+1}e^{-1/2y^2}}{\Gamma[(n-2)/2]2^{(n-4)/2}}dy.$$

The integral is equal to one, and so

$$E\left[Y^2\right] = E\left[\left(\chi_n^{-1}\right)^2\right] = \frac{\Gamma[(n-2)/2]2^{(n-4)/2}}{\Gamma[n/2]2^{(n-2)/2}} = \frac{1}{2}\frac{\Gamma[(n-2)/2]}{\Gamma[n/2]} = \frac{1}{n-2}, \tag{3.9}$$

which is finite for real n greater than two.

The variance of Y is seen to be

$$Var[Y] = Var\left[\chi_n^{-1}\right] = E\left[Y^2\right] - E^2[Y]$$

$$= \frac{1}{n-2} - \left\{ \frac{\Gamma[(n-1)/2]}{\Gamma[n/2]} \frac{1}{\sqrt{2}} \right\}^2 = \frac{1}{n-2} - \frac{1}{2} \frac{\Gamma^2[(n-1)/2]}{\Gamma^2[n/2]}, \tag{3.10}$$

which is finite for real n greater than two.

Note that each of the mean and variance of the χ_n^2 random variable grows without bound as n grows without bound. On the other hand, each of the mean and variance of the χ_n^{-1} random variable approaches zero as n grows without bound. The χ_n^2 runs away to infinity while the χ_n^{-1} collapses into the origin.

Now consider again the random variable $\hat{C}p/Cp$. Recall that $\hat{C}p/Cp$ is distributed as $\sqrt{n}\,\chi_n^{-1}$, that is, the square root of n times an inverted chi with n degrees of freedom. We have

$$E\left[\frac{\hat{C}p}{Cp}\right] = E\left[\sqrt{n}\,\chi_n^{-1}\right] = \sqrt{n}E\left[\chi_n^{-1}\right] = \frac{\Gamma[(n-1)/2]}{\Gamma[n/2]}\sqrt{\frac{n}{2}} \tag{3.11}$$

and

$$Var\left[\frac{\hat{C}p}{Cp}\right] = Var\left[\sqrt{n}\,\chi_n^{-1}\right] = nVar\left[\chi_n^{-1}\right] = \frac{n}{n-2} - \frac{n}{2}\frac{\Gamma^2[(n-1)/2]}{\Gamma^2[n/2]}$$

$$= \frac{n}{n-2} - \left\{ \frac{\Gamma[(n-1)/2]}{\Gamma[n/2]}\sqrt{\frac{n}{2}} \right\}^2 = \frac{n}{n-2} - E^2\left[\frac{\hat{C}p}{Cp}\right]. \tag{3.12}$$

From Abramowitz *et al.* (1965), page 257,

$$\lim_{n\to\infty} n^{b-a}\frac{\Gamma[n+a]}{\Gamma[n+b]} = 1 \tag{3.13}$$

for any real a and b, and so we see that $E\left[\hat{C}p/Cp\right]$ approaches one, while $Var\left[\hat{C}p/Cp\right]$ approaches zero, as n grows large. Therefore, if we take the mean squared error of the random variable $\hat{C}p/Cp$ around one, we get

$$MSE\left[\frac{\hat{C}p}{Cp}\right] = E\left[\left\{\frac{\hat{C}p}{Cp} - 1\right\}^2\right] = Var\left[\frac{\hat{C}p}{Cp}\right] + \left\{E\left[\frac{\hat{C}p}{Cp}\right] - 1\right\}^2$$

$$= \frac{n}{n-2} - E^2\left[\frac{\hat{C}p}{Cp}\right] + \left\{E\left[\frac{\hat{C}p}{Cp}\right] - 1\right\}^2, \tag{3.14}$$

which approaches zero as n grows large. Therefore, the statistic $\hat{C}p$ is consistent both in mean square and in probability for Cp. Though it has a positive bias in finite samples, the statistic $\hat{C}p$ is asymptotically unbiased for Cp. The relative bias can be viewed in Table 3.1, which we address shortly.

It will be instructive to approximate the mean and variance of $\hat{C}p/Cp$ by finite Taylor series expansion even though we have exact results in this simplest of cases. Doing so will help us evaluate the accuracy of these Taylor series approximations. We will not have this luxury when we later consider autocorrelated data. At that point, we must rely on simulations to assess our approximations.

Let g be a twice-differentiable, real-valued function. Then $E[g(Y)]$ is approximately given by

$$E[g(Y)] \approx gE[Y] + \frac{1}{2}g''E[Y] \cdot Var[Y]. \tag{3.15}$$

Let $Y = \sum_{i=1}^{n}(X_i - \mu)^2$. We know that

$$E\left[\frac{1}{\sigma^2}Y\right] = n \quad \text{and} \quad Var\left[\frac{1}{\sigma^2}Y\right] = 2n.$$

Therefore,

$$E[Y] = n\sigma^2 \quad \text{and} \quad Var[Y] = 2n\sigma^4.$$

Letting $g(Y) = \frac{1}{\sqrt{Y}} = Y^{-1/2}$, $g'(Y) = -\frac{1}{2}Y^{-3/2}$, and $g''(Y) = \frac{3}{4}Y^{-5/2}$, we have

$$E[g(Y)] \approx gE[Y] + \frac{1}{2}g''E[Y] \cdot Var[Y]$$
$$= \left(n\sigma^2\right)^{-1/2} + \frac{1}{2}\frac{3}{4}\left(n\sigma^2\right)^{-5/2} \cdot 2n\sigma^4$$

$$= \frac{1}{\sigma\sqrt{n}} + \frac{3}{4}\frac{1}{\sigma n^{3/2}},$$

and so

$$E\left[\frac{\hat{C}p}{Cp}\right] = E\left[\frac{\sigma}{S}\right] = \sigma\sqrt{n}E\left[\frac{1}{\sqrt{Y}}\right] = \sigma\sqrt{n}E\left[g(Y)\right]$$

$$\approx \sigma\sqrt{n}\left\{\frac{1}{\sigma\sqrt{n}} + \frac{3}{4}\frac{1}{\sigma n^{3/2}}\right\}$$

$$= 1 + \frac{3}{4n}.$$

That is,

$$E\left[\frac{\hat{C}p}{Cp}\right] \approx \hat{E}\left[\frac{\hat{C}p}{Cp}\right] = 1 + \frac{3}{4n}. \tag{3.16}$$

How accurate is this approximation? Table 3.1 allows comparison of this approximation with the true mean for various sample sizes n. Let $\hat{\theta} = \hat{C}p/Cp$. Also, let $E\left[\hat{\theta}\right]$ be the true mean of $\hat{\theta} = \hat{C}p/Cp$ and let $\hat{E}\left[\hat{\theta}\right]$ be the Taylor series approximation of the mean of $\hat{\theta} = \hat{C}p/Cp$. We denote the relative error as $\left(E - \hat{E}\right)/E$. Repeating, we have

$$E\left[\hat{\theta}\right] = E\left[\frac{\hat{C}p}{Cp}\right] = \frac{\Gamma\left[(n-1)/2\right]}{\Gamma\left[n/2\right]}\sqrt{\frac{n}{2}}, \tag{3.17}$$

$$\hat{E}\left[\hat{\theta}\right] = \hat{E}\left[\frac{\hat{C}p}{Cp}\right] = 1 + \frac{3}{4n}, \tag{3.18}$$

$$\left(E - \hat{E}\right)/E = \frac{E\left[\hat{\theta}\right] - \hat{E}\left[\hat{\theta}\right]}{E\left[\hat{\theta}\right]} = 1 - \frac{\hat{E}\left[\hat{\theta}\right]}{E\left[\hat{\theta}\right]}. \tag{3.19}$$

We see from Table 3.1 that the approximation is remarkably accurate. Its relative error is less than four percent for n as small as five, and less than one percent for n as small as ten.

Next we approximate $Var\left[\hat{C}p/Cp\right]$ by the Taylor series approximation for the variance of a function of a random variable,

Table 3.1. Exact vs. Approximate Mean of $\hat{\theta} = \hat{C}p/Cp$

n	$E[\hat{\theta}]$	$\hat{E}[\hat{\theta}]$	$(E-\hat{E})/E$
5	1.18942	1.15000	0.03314
10	1.08379	1.07500	0.00811
20	1.03956	1.03750	0.00198
30	1.02590	1.02500	0.00088
40	1.01925	1.01875	0.00049
50	1.01532	1.01500	0.00032
100	1.00758	1.00750	0.00008
200	1.00377	1.00375	0.00002
300	1.00251	1.00250	0.00001
400	1.00188	1.00188	0.00000
500	1.00150	1.00150	0.00000
1000	1.00075	1.00075	0.00000

$$Var[g(Y)] \approx \{g'E[Y]\}^2 \cdot Var[Y]. \tag{3.20}$$

Let $Y = \sum_{i=1}^{n}(X_i - \mu)^2$. We know that $E[Y] = n\sigma^2$ and $Var[Y] = 2n\sigma^4$. Letting $g(Y) = \frac{1}{\sqrt{Y}} = Y^{-1/2}$, $g'(Y) = -\frac{1}{2}Y^{-3/2}$, $g''(Y) = \frac{3}{4}Y^{-5/2}$, we have

$$\begin{aligned} Var[g(Y)] &\approx \{g'E[Y]\}^2 \cdot Var[Y] \\ &= \left\{-\frac{1}{2}\left(n\sigma^2\right)^{-3/2}\right\}^2 \cdot 2n\sigma^4 \\ &= \frac{1}{2n^2\sigma^2}, \end{aligned}$$

and so

$$\begin{aligned} Var\left[\frac{\hat{C}p}{Cp}\right] &= Var\left[\frac{\sigma}{S}\right] = n\sigma^2 Var\left[\frac{1}{\sqrt{Y}}\right] = n\sigma^2 Var[g(Y)] \\ &\approx \frac{n\sigma^2}{2n^2\sigma^2} = \frac{1}{2n}. \end{aligned}$$

That is,

$$Var\left[\frac{\hat{C}p}{Cp}\right] \approx \hat{V}ar\left[\frac{\hat{C}p}{Cp}\right] = \frac{1}{2n}. \tag{3.21}$$

The mean squared error of $\hat{C}p/Cp$ around one is approximately given by

$$MSE\left[\frac{\hat{C}p}{Cp}\right] = E\left[\left\{\frac{\hat{C}p}{Cp} - 1\right\}^2\right] = Var\left[\frac{\hat{C}p}{Cp}\right] + \left\{E\left[\frac{\hat{C}p}{Cp}\right] - 1\right\}^2$$

$$\approx \hat{V}ar\left[\frac{\hat{C}p}{Cp}\right] + \left\{\hat{E}\left[\frac{\hat{C}p}{Cp}\right] - 1\right\}^2$$

$$= \frac{1}{2n} + \left\{1 + \frac{3}{4n} - 1\right\}^2 = \frac{1}{2n} + \left\{\frac{3}{4n}\right\}^2, \tag{3.22}$$

which approaches zero as n gets large, showing that the sequence of approximating statistics is consistent both in mean square and in probability for Cp.

The chi-squared random variable has one parameter, n. Consequently, its mean and variance are functionally dependent. We could use this fact to directly insert our Taylor series-approximated $E\left[\hat{C}p/Cp\right]$ into the $Var\left[\hat{C}p/Cp\right]$. Substituting equation (3.16) into equation (3.12), we have

$$Var\left[\frac{\hat{C}p}{Cp}\right] = \frac{n}{n-2} - E^2\left[\frac{\hat{C}p}{Cp}\right] \approx \frac{n}{n-2} - \hat{E}^2\left[\frac{\hat{C}p}{Cp}\right] = \frac{n}{n-2} - \left\{1 + \frac{3}{4n}\right\}^2,$$

that is,

$$Var\left[\frac{\hat{C}p}{Cp}\right] \approx \tilde{V}ar\left[\frac{\hat{C}p}{Cp}\right] = \frac{n}{n-2} - \left\{1 + \frac{3}{4n}\right\}^2. \tag{3.23}$$

We see that $\hat{E}\left[\hat{C}p/Cp\right]$ approaches one, while $\tilde{V}ar\left[\hat{C}p/Cp\right]$ approaches zero, as n grows large, showing that the sequence of approximating statistics is consistent both in mean square and in probability for Cp by this alternate approximation. Of course, we must remark that consistency is the minimum requirement of any reasonable estimator.

We can evaluate our two approaches to approximating $Var\left[\hat{C}p/Cp\right]$. Let $\hat{\theta} = \hat{C}p/Cp$. Also, let $Var\left[\hat{\theta}\right]$ be the true variance of $\hat{\theta} = \hat{C}p/Cp$ and let $\hat{Var}\left[\hat{\theta}\right]$ be the estimated variance of $\hat{\theta} = \hat{C}p/Cp$ computed directly from the Taylor series approximation for the variance of $\hat{\theta} = \hat{C}p/Cp$. Finally, let $\tilde{Var}\left[\hat{\theta}\right]$ be the estimated variance of $\hat{\theta} = \hat{C}p/Cp$ computed recursively from the Taylor series approximation for the mean of $\hat{\theta} = \hat{C}p/Cp$. We denote the relative errors as $\left(V - \hat{V}\right)/V$ and $\left(V - \tilde{V}\right)/V$. Repeating, we have

$$\hat{\theta} = \frac{\hat{C}p}{Cp}, \tag{3.24}$$

$$E\left[\hat{\theta}\right] = E\left[\frac{\hat{C}p}{Cp}\right] = \frac{\Gamma[(n-1)/2]}{\Gamma[n/2]}\sqrt{\frac{n}{2}}, \tag{3.25}$$

$$Var\left[\hat{\theta}\right] = Var\left[\frac{\hat{C}p}{Cp}\right] = \frac{n}{n-2} - E^2\left[\hat{\theta}\right] = \frac{n}{n-2} - \left\{\frac{\Gamma[(n-1)/2]}{\Gamma[n/2]}\sqrt{\frac{n}{2}}\right\}^2, \tag{3.26}$$

$$\hat{E}\left[\hat{\theta}\right] = \hat{E}\left[\frac{\hat{C}p}{Cp}\right] = 1 + \frac{3}{4n}, \tag{3.27}$$

$$\hat{Var}\left[\hat{\theta}\right] = \hat{Var}\left[\frac{\hat{C}p}{Cp}\right] = \frac{1}{2n}, \tag{3.28}$$

$$\tilde{Var}\left[\hat{\theta}\right] = \tilde{Var}\left[\frac{\hat{C}p}{Cp}\right] = \frac{n}{n-2} - \hat{E}^2\left[\hat{\theta}\right] = \frac{n}{n-2} - \left\{1 + \frac{3}{4n}\right\}^2, \tag{3.29}$$

$$\left(V - \hat{V}\right)/V = \frac{Var\left[\hat{\theta}\right] - \hat{Var}\left[\hat{\theta}\right]}{Var\left[\hat{\theta}\right]} = 1 - \frac{\hat{Var}\left[\hat{\theta}\right]}{Var\left[\hat{\theta}\right]}, \tag{3.30}$$

$$\left(V - \tilde{V}\right)/V = \frac{Var\left[\hat{\theta}\right] - \tilde{Var}\left[\hat{\theta}\right]}{Var\left[\hat{\theta}\right]} = 1 - \frac{\tilde{Var}\left[\hat{\theta}\right]}{Var\left[\hat{\theta}\right]}. \tag{3.31}$$

Table 3.2 reveals that the Taylor series estimator $\hat{Var}\left[\hat{\theta}\right]$ consistently underestimates the true variance while the recursive estimator $\tilde{Var}\left[\hat{\theta}\right]$ overestimates the true variance. It takes a sample size of $n = 100$ before the relative error of each falls below four percent.

Table 3.2. Exact vs. Two Approximate Variances of $\hat{\theta} = \hat{C}p/Cp$

n	$Var[\hat{\theta}]$	$\hat{V}ar[\hat{\theta}]$	$(V-\hat{V})/V$	$Var[\hat{\theta}]$	$\tilde{V}ar[\hat{\theta}]$	$(V-\tilde{V})/V$
5	0.251956	0.100000	0.603105	0.251956	0.344167	-0.365981
10	0.075546	0.050000	0.338152	0.075546	0.094375	-0.249239
20	0.030424	0.025000	0.178280	0.030424	0.034705	-0.140711
30	0.018959	0.016667	0.120893	0.018959	0.020804	-0.097315
40	0.013758	0.012500	0.091438	0.013758	0.014780	-0.074284
50	0.010794	0.010000	0.073559	0.010794	0.011442	-0.060033
100	0.005193	0.005000	0.037165	0.005193	0.005352	-0.030618
200	0.002548	0.002500	0.018838	0.002548	0.002587	-0.015306
300	0.001688	0.001667	0.012441	0.001688	0.001705	-0.010071
400	0.001262	0.001250	0.009509	0.001262	0.001272	-0.007924
500	0.001008	0.001000	0.007936	0.001008	0.001014	-0.005952
1000	0.000502	0.000500	0.003984	0.000502	0.000503	-0.001992

When the population standard deviation σ is unknown, the true Cp is unknown, hence the estimator of Cp is subject to sampling error. When the sampled data are *iid* normal, it is a simple matter to report a lower confidence bound for the true Cp. Since $\hat{C}p/Cp = \sigma/S$ is distributed as $\sqrt{n}\,\chi_n^{-1}$, the $(1 - \alpha)$ lower confidence bound for the true Cp is given by $\hat{C}p\sqrt{\dfrac{\chi^2_{n,\alpha}}{n}}$, where $\chi^2_{n,\alpha}$ is the (lower) α percentile of χ^2_n. Table 3.3 gives the 0.95 lower confidence bound for the true Cp for a selected sample size n and a selected value of $\hat{C}p = \dfrac{USL - LSL}{6S}$, where $S = \sqrt{\dfrac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2}$. For example, when Cp is estimated to be 1.40 using a sample size of $n = 20$, the 0.95 lower confidence bound for the true Cp is given by 1.03120. That is, one is 0.95 confident that the true Cp is at least 1.03120. Yet when Cp is estimated to be 1.40 using a sample size of $n = 1000$, one is 0.95 confident that the true Cp is at least 1.34751. We see that increasing the sample size n increases the precision of the lower bound, as it should.

Table 3.3. 0.95 Lower Confidence Bound on True Cp with Known μ

	Estimated Cp				
n	1.00	1.20	1.40	1.60	1.80
5	0.47864	0.57437	0.67010	0.76582	0.86155
10	0.62772	0.75326	0.87881	1.00435	1.12990
20	0.73657	0.88388	1.03120	1.17851	1.32583
30	0.78512	0.94214	1.09917	1.25619	1.41322
40	0.81408	0.97690	1.13971	1.30253	1.46534
50	0.83384	1.00061	1.16738	1.33414	1.50091
100	0.88278	1.05934	1.23589	1.41245	1.58900
200	0.91728	1.10074	1.28419	1.46765	1.65110
300	0.93252	1.11902	1.30553	1.49203	1.67854
400	0.94160	1.12992	1.31824	1.50656	1.69488
500	0.94655	1.13586	1.32517	1.51448	1.70379
1000	0.96251	1.15501	1.34751	1.54002	1.73252

We note that this lower bound has a Bayesian interpretation in that it corresponds to the lower α percentile of the posterior distribution of $Cp/\hat{C}p = S/\sigma = \left\{\frac{1}{n\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2\right\}^{1/2}$ when μ is known and the prior density of σ is taken to be the standard noninformative prior on the positive reals, that is, σ proportional to $1/\sigma$ on the positive reals or equivalently, $\log_e\sigma$ proportional to a constant over the reals. For example, when the estimated Cp is 1.40, the posterior probability is 0.95 that the true Cp is at least 1.03120. See Box and Tiao (1973) and Zellner (1971) for excellent discussions on noninformative prior distributions.

Let π_0 be the proportion of X outside the specification interval $[LSL, USL]$ and let π_1 be the proportion of X within, so that $\pi_0 + \pi_1 = 1$. If $\mu = m$, then $Cpl = Cp = Cpu$ and so

$$\begin{aligned}\pi_0 &= 1-\pi_1 = 1-\Pr[LSL \le X \le USL]\\ &= 1-\Pr[-3Cpl \le Z \le 3Cpu]\\ &= 1-\Pr[-3Cp \le Z \le 3Cp]\end{aligned}$$

Table 3.4. 0.95 Upper Confidence Bound on True π_0 with Known μ

n	Estimated Cp 1.00	1.20	1.40	1.60	1.80
5	0.15102	0.08487	0.04440	0.02159	0.00975
10	0.05968	0.02383	0.00838	0.00259	0.00070
20	0.02712	0.00801	0.00198	0.00041	0.00007
30	0.01850	0.00471	0.00098	0.00016	0.00002
40	0.01460	0.00338	0.00063	0.00009	0.00001
50	0.01237	0.00268	0.00046	0.00006	0.00001
100	0.00809	0.00148	0.00021	0.00002	0.00001
200	0.00593	0.00096	0.00012	0.00001	0.00001
300	0.00515	0.00079	0.00009	0.00001	0.00001
400	0.00473	0.00070	0.00008	0.00001	0.00001
500	0.00452	0.00066	0.00007	0.00001	0.00001
1000	0.00388	0.00053	0.00005	0.00001	0.00001

$$= 1 - \{\Phi[3Cp] - \Phi[-3Cp]\}$$

$$= 1 - \{1 - \Phi[-3Cp] - \Phi[-3Cp]\}$$

$$= 2\Phi[-3Cp], \tag{3.32}$$

where Φ is the standard normal cumulative distribution function. Now since Φ is strictly increasing in its argument, the (1 - α) upper confidence bound for the true π_0 is given by

$2\Phi\left[-3\hat{C}p\sqrt{\frac{\chi^2_{n,\alpha}}{n}}\right]$, where $\hat{C}p\sqrt{\frac{\chi^2_{n,\alpha}}{n}}$ is the (1 - α) lower confidence bound for the true Cp.

Table 3.4 gives the 0.95 upper confidence bound for the true π_0 for a selected sample size n and a selected value of $\hat{C}p = \frac{USL - LSL}{6S}$. For example, when Cp is estimated to be 1.40 using a sample size of $n = 20$, the 0.95 upper confidence bound for the true π_0 is given by 0.00198. That is, one is 0.95 confident that the true π_0 is no greater than 0.00198. Even if the process mean μ is not located at the specification interval midpoint m, we can still interpret this confidence bound as a measure of the *potential* π_0, subject to centering the process in the specification interval without disturbing σ.

3.3. Estimating Cp when Both μ and σ are Unknown

When both μ and σ are unknown, we have

$$\frac{\hat{C}p}{Cp} = \frac{(USL - LSL)/6S}{(USL - LSL)/6\sigma} = \frac{\sigma}{S}, \tag{3.33}$$

where here

$$S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2}.$$

It is well known that for $\{X_i\}_n$ distributed *iid* $N(\mu, \sigma^2)$, the random variable $\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \bar{X})^2$ is distributed as χ^2_{n-1} , a chi-squared random variable with (n - 1) degrees of freedom. It follows that the random variable $\hat{C}p/Cp = \sigma/S$ is distributed as $\sqrt{n-1}\,\chi^{-1}_{n-1}$, the square root of ($n$ - 1) times an "inverted chi" with (n - 1) degrees of freedom. Its mean and variance are

$$E\left[\frac{\hat{C}p}{Cp}\right] = E\left[\sqrt{n-1}\,\chi^{-1}_{n-1}\right] = \sqrt{n-1}\,E\left[\chi^{-1}_{n-1}\right] = \frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}} \tag{3.34}$$

and

$$Var\left[\frac{\hat{C}p}{Cp}\right] = Var\left[\sqrt{n-1}\,\chi^{-1}_{n-1}\right] = (n-1)\,Var\left[\chi^{-1}_{n-1}\right] = \frac{n-1}{n-3} - \frac{n-1}{2}\frac{\Gamma^2[(n-2)/2]}{\Gamma^2[(n-1)/2]}$$

$$= \frac{n-1}{n-3} - \left\{\frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}}\right\}^2 = \frac{n-1}{n-3} - E^2\left[\frac{\hat{C}p}{Cp}\right]. \tag{3.35}$$

We see that $E\left[\hat{C}p/Cp\right]$ approaches one, while $Var\left[\hat{C}p/Cp\right]$ approaches zero, as n grows large. Therefore, the statistic $\hat{C}p$ is consistent both in mean square and in probability for Cp. Though it has a positive bias in finite samples, the statistic $\hat{C}p$ is asymptotically unbiased for Cp. The relative bias can be viewed in Table 3.1, if one adjusts for the loss of one degree of freedom.

It is interesting to note that if we use the sample mean instead of μ in the calculation of $\hat{C}p$, that is, if we use $S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2}$ rather than $\sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2}$, we get

$$E\left[\frac{\hat{C}p}{Cp}\right] = \frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}} \approx 1 + \frac{3}{4(n-1)} = 1 + \frac{3}{4n} + \frac{3}{4n(n-1)} \tag{3.36}$$

and

$$Var\left[\frac{\hat{C}p}{Cp}\right] = \frac{n-1}{n-3} - \left\{\frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}}\right\}^2 \approx \frac{1}{2(n-1)} = \frac{1}{2n} + \frac{1}{2n(n-1)}. \tag{3.37}$$

The additional terms, $\frac{3}{4n(n-1)}$ and $\frac{1}{2n(n-1)}$, are estimates of the pure or incremental bias in each estimator due to the use of the sample mean, instead of the population mean, in calculating S. They represent the cost of not knowing the population mean μ.

The mean squared error of $\hat{C}p/Cp$ around one is approximated as

$$\begin{aligned} MSE\left[\frac{\hat{C}p}{Cp}\right] &= Var\left[\frac{\hat{C}p}{Cp}\right] + \left\{E\left[\frac{\hat{C}p}{Cp}\right] - 1\right\}^2 \\ &= \frac{n-1}{n-3} - E^2\left[\frac{\hat{C}p}{Cp}\right] + \left\{E\left[\frac{\hat{C}p}{Cp}\right] - 1\right\}^2 \\ &\approx \frac{n-1}{n-3} - \hat{E}^2\left[\frac{\hat{C}p}{Cp}\right] + \left\{\hat{E}\left[\frac{\hat{C}p}{Cp}\right] - 1\right\}^2 \\ &= \frac{n-1}{n-3} - \left\{1 + \frac{3}{4(n-1)}\right\}^2 + \left\{1 + \frac{3}{4(n-1)} - 1\right\}^2, \end{aligned} \tag{3.38}$$

which approaches zero as n gets large, showing that $\hat{C}p$ is consistent both in mean square and in probability for Cp in its Taylor series approximation.

Table 3.5. 0.95 Lower Confidence Bound on True *Cp* with Unknown μ

n	Estimated Cp 1.00	1.20	1.40	1.60	1.80
5	0.42152	0.50583	0.59013	0.67443	0.75874
10	0.60783	0.72940	0.85096	0.97253	1.09409
20	0.72971	0.87560	1.02159	1.16753	1.31347
30	0.78143	0.93771	1.09400	1.25029	1.40657
40	0.81170	0.97404	1.13638	1.29872	1.46106
50	0.83214	0.99857	1.16499	1.33142	1.49785
100	0.88218	1.05862	1.23506	1.41149	1.58793
200	0.91707	1.10048	1.28389	1.46731	1.65072
300	0.93241	1.11889	1.30537	1.49185	1.67833
400	0.94152	1.12983	1.31813	1.50644	1.69474
500	0.94650	1.13580	1.32510	1.51440	1.70369
1000	0.96249	1.15499	1.34749	1.53999	1.73249

Of course if $S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2}$ is used in the computation of $\hat{C}p$, then we have $(n - 1)$ degrees of freedom and the $(1 - \alpha)$ lower confidence bound for the true Cp is given by $\hat{C}p\sqrt{\frac{\chi^2_{n-1,\alpha}}{n-1}}$, where $\chi^2_{n-1,\alpha}$ is the (lower) α percentile of χ^2_{n-1}. Table 3.5 gives the 0.95 lower confidence bound for the true Cp for a selected sample size n and a selected value of $\hat{C}p = \frac{USL - LSL}{6S}$, where $S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2}$. For example, when Cp is estimated to be 1.40 using a sample size of $n = 20$, the 0.95 lower confidence bound for the true Cp is given by 1.02159. That is, one is 0.95 confident that the true Cp is at least 1.02159.

Again we observe that this lower bound has a Bayesian interpretation in that it corresponds to the lower α percentile of the posterior distribution of $Cp/\hat{C}p = S/\sigma = \left\{\frac{1}{(n-1)\sigma^2}\sum_{i=1}^{n}(X_i - \bar{X})^2\right\}^{1/2}$ when the prior density of (μ, σ) is taken to be

Table 3.6. 0.95 Upper Confidence Bound on True π_0 with Unknown μ

n	Estimated Cp 1.00	1.20	1.40	1.60	1.80
5	0.20603	0.12914	0.07666	0.04304	0.02283
10	0.06823	0.02866	0.01068	0.00353	0.00103
20	0.02859	0.00862	0.00218	0.00046	0.00008
30	0.01906	0.00491	0.00103	0.00017	0.00002
40	0.01489	0.00348	0.00065	0.00010	0.00001
50	0.01255	0.00274	0.00047	0.00007	0.00001
100	0.00813	0.00149	0.00021	0.00002	0.00001
200	0.00594	0.00096	0.00012	0.00001	0.00001
300	0.00515	0.00079	0.00009	0.00001	0.00001
400	0.00474	0.00070	0.00008	0.00001	0.00001
500	0.00452	0.00066	0.00007	0.00001	0.00001
1000	0.00388	0.00053	0.00005	0.00001	0.00001

the standard noninformative prior, that is, σ proportional to $1/\sigma$ on the positive reals and independently, μ proportional to a constant over the reals.

If $\mu = m$, then the $(1 - \alpha)$ upper confidence bound for the true π_0 is given by $2\Phi\left[-3\hat{C}p\sqrt{\frac{\chi^2_{n-1}}{n-1}}\right]$, where $\hat{C}p\sqrt{\frac{\chi^2_{n-1}}{n-1}}$ is the $(1 - \alpha)$ lower confidence bound for the true Cp. Table 3.6 gives the 0.95 upper confidence bound for the true π_0 for a selected sample size n and a selected value of $\hat{C}p = \frac{USL - LSL}{6S}$. For example, when Cp is estimated to be 1.40 using a sample size of $n = 20$, the 0.95 upper confidence bound for the true π_0 is given by 0.00218. That is, one is 0.95 confident that the true π_0 is no greater than 0.00218. Again, even if the process mean μ is not located at the specification interval midpoint m, we can still view this confidence bound as a measure of the *potential* π_0, subject to centering the process in the specification interval without disturbing σ. And again, this might require no more than a simple adjustment by the machine operator.

3.4. Estimating (Cpl, Cp, Cpu) when μ is Unknown and σ is Known

From the definitions

$$(Cpl, Cp, Cpu) = \left(\frac{\mu - LSL}{3\sigma}, \frac{USL - LSL}{6\sigma}, \frac{USL - \mu}{3\sigma} \right), \tag{3.39}$$

we see the basic identity

$$Cp = \frac{1}{2}(Cpl + Cpu). \tag{3.40}$$

An important observation is that for *any* joint estimators of the form

$$\left(\hat{C}pl, \hat{C}p, \hat{C}pu\right) = \left(\frac{\hat{\mu} - LSL}{3\hat{\sigma}}, \frac{USL - LSL}{6\hat{\sigma}}, \frac{USL - \hat{\mu}}{3\hat{\sigma}} \right), \tag{3.41}$$

a similar identity obtains, namely

$$\hat{C}p = \frac{1}{2}\left(\hat{C}pl + \hat{C}pu\right). \tag{3.42}$$

Now consider the natural estimators

$$\left(\hat{C}pl, \hat{C}p, \hat{C}pu\right) = \left(\frac{\overline{X} - LSL}{3\sigma}, \frac{USL - LSL}{6\sigma}, \frac{USL - \overline{X}}{3\sigma} \right), \tag{3.43}$$

where $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Note that here, $\hat{C}p = Cp$ is a known constant.

We have

$$E\left[\hat{C}pl\right] = E\left[\frac{\overline{X} - LSL}{3\sigma}\right] = \frac{\mu - LSL}{3\sigma} = Cpl \tag{3.44}$$

and

$$\begin{aligned} Var\left[\hat{C}pl\right] &= Var\left[\frac{\overline{X} - LSL}{3\sigma}\right] = \frac{1}{9\sigma^2} Var\left[\overline{X} - LSL\right] \\ &= \frac{1}{9\sigma^2} Var\left[\overline{X}\right] = \frac{1}{9\sigma^2}\frac{\sigma^2}{n} = \frac{1}{9n}. \end{aligned} \tag{3.45}$$

We also have

$$E\left[\hat{C}pu\right]=E\left[\frac{USL-\overline{X}}{3\sigma}\right]=\frac{USL-\mu}{3\sigma}=Cpu \tag{3.46}$$

and

$$Var\left[\hat{C}pu\right]=Var\left[\frac{USL-\overline{X}}{3\sigma}\right]=\frac{1}{9\sigma^2}Var\left[USL-\overline{X}\right]$$

$$=\frac{1}{9\sigma^2}Var\left[\overline{X}\right]=\frac{1}{9\sigma^2}\frac{\sigma^2}{n}=\frac{1}{9n}. \tag{3.47}$$

The covariance can be gotten as

$$Cov\left[\hat{C}pl,\hat{C}pu\right]=Cov\left[\frac{\overline{X}-LSL}{3\sigma},\frac{USL-\overline{X}}{3\sigma}\right]=\frac{1}{9\sigma^2}Cov\left[\overline{X}-LSL,USL-\overline{X}\right]$$

$$=\frac{1}{9\sigma^2}Cov\left[\overline{X},-\overline{X}\right]=-\frac{1}{9\sigma^2}Cov\left[\overline{X},\overline{X}\right]=-\frac{1}{9\sigma^2}Var\left[\overline{X}\right]$$

$$=-\frac{1}{9\sigma^2}\frac{\sigma^2}{n}=-\frac{1}{9n}. \tag{3.48}$$

Of course, we have

$$Corr\left[\hat{C}pl,\hat{C}pu\right]=\frac{Cov\left[\hat{C}pl,\hat{C}pu\right]}{\sqrt{Var\left[\hat{C}pl\right]}\sqrt{Var\left[\hat{C}pu\right]}}=\frac{-1/9n}{\sqrt{1/9n}\sqrt{1/9n}}=-1. \tag{3.49}$$

Now since $\hat{C}p=Cp$ is a known constant, we should observe

$$E\left[\hat{C}p\right]=E[Cp]=Cp \quad \text{and} \quad Var\left[\hat{C}p\right]=Var[Cp]=0, \tag{3.50}$$

which we confirm by

$$E\left[\hat{C}p\right]=E\left[\frac{1}{2}\left(\hat{C}pl+\hat{C}pu\right)\right]=\frac{1}{2}E\left[\hat{C}pl\right]+\frac{1}{2}E\left[\hat{C}pu\right]$$

$$=\frac{1}{2}Cpl+\frac{1}{2}Cpu=Cp \tag{3.51}$$

and

$$Var\left[\hat{C}p\right]=Var\left[\frac{1}{2}\left(\hat{C}pl+\hat{C}pu\right)\right]$$

$$=\frac{1}{4}Var\left[\hat{C}pl\right]+\frac{1}{4}Var\left[\hat{C}pu\right]+\frac{2}{4}Cov\left[\hat{C}pl,\hat{C}pu\right]$$

$$=\frac{1}{4}\frac{1}{9n}+\frac{1}{4}\frac{1}{9n}+\frac{2}{4}\left(-\frac{1}{9n}\right)=0. \tag{3.52}$$

In fact, each of $\hat{C}pl$ and $\hat{C}pu$ is normal, each being a linear function of the normal $\bar{X}$. Yet the pair $(\hat{C}pl, \hat{C}pu)$ is degenerate in the line $\frac{1}{2}\hat{C}pl+\frac{1}{2}\hat{C}pu = Cp$. This means that either of the pair $(\hat{C}pl, \hat{C}pu)$ can be estimated and we automatically get joint power, precision, and confidence for both.

With μ unknown and σ known, the statistic

$$(\hat{C}pl, \hat{C}p, \hat{C}pu)=\left(\frac{\bar{X}-LSL}{3\sigma},\ \frac{USL-LSL}{6\sigma},\ \frac{USL-\bar{X}}{3\sigma}\right)$$

has one degree of freedom represented by $\bar{X}$. It is both the maximum likelihood (ML) estimator and the uniformly minimum variance unbiased (UMVU) estimator for the population triple index (Cpl, Cp, Cpu). Since $\hat{C}pl$ is normal with mean Cpl and variance $1/9n$, we know that a $(1 - \alpha)$ confidence interval for the true Cpl is given by

$$\left[\hat{C}pl - z_{\alpha/2}\sqrt{\frac{1}{9n}},\ \hat{C}pl + z_{\alpha/2}\sqrt{\frac{1}{9n}}\right], \tag{3.53}$$

where $z_{\alpha/2}$ is the upper $\alpha/2$ percentile of the standard normal random variable. Similarly, a $(1 - \alpha)$ confidence interval for the true Cpu is given by

$$\left[\hat{C}pu - z_{\alpha/2}\sqrt{\frac{1}{9n}},\ \hat{C}pu + z_{\alpha/2}\sqrt{\frac{1}{9n}}\right]. \tag{3.54}$$

A joint $(1 - \alpha)$ confidence region for the true (Cpl, Cpu) is given by the line segment joining the points in (Cpl, Cpu)-space,

$$\left(\hat{C}pl - z_{\alpha/2}\sqrt{\frac{1}{9n}},\ \hat{C}pu + z_{\alpha/2}\sqrt{\frac{1}{9n}}\right)$$

$$\text{and}\quad \left(\hat{C}pl + z_{\alpha/2}\sqrt{\frac{1}{9n}},\ \hat{C}pu - z_{\alpha/2}\sqrt{\frac{1}{9n}}\right). \tag{3.55}$$

In other words, a joint $(1 - \alpha)$ confidence region for the true (Cpl, Cpu) is given by the line segment in the (Cpl, Cpu) plane,

$$\gamma\left(\hat{C}pl - z_{\alpha/2}\sqrt{\frac{1}{9n}},\quad \hat{C}pu + z_{\alpha/2}\sqrt{\frac{1}{9n}}\right)$$
$$+(1-\gamma)\left(\hat{C}pl + z_{\alpha/2}\sqrt{\frac{1}{9n}},\quad \hat{C}pu - z_{\alpha/2}\sqrt{\frac{1}{9n}}\right) \tag{3.56}$$

for all real $0 \le \gamma \le 1$. Also, a joint $(1 - \alpha)$ confidence region for the true (Cpl, Cp, Cpu) is given by the line segment in (Cpl, Cp, Cpu)-space,

$$\gamma\left(\hat{C}pl - z_{\alpha/2}\sqrt{\frac{1}{9n}},\quad \hat{C}p,\quad \hat{C}pu + z_{\alpha/2}\sqrt{\frac{1}{9n}}\right)$$
$$+(1-\gamma)\left(\hat{C}pl + z_{\alpha/2}\sqrt{\frac{1}{9n}},\quad \hat{C}p,\quad \hat{C}pu - z_{\alpha/2}\sqrt{\frac{1}{9n}}\right) \tag{3.57}$$

for all real $0 \le \gamma \le 1$. Recall that with σ known, $\hat{C}p = Cp$, a known constant.

To take a numerical example, suppose we have estimated

$$\left(\hat{C}pl,\quad \hat{C}p,\quad \hat{C}pu\right) = (1.200,\quad 1.250,\quad 1.300) \tag{3.58}$$

based on $n = 100$. For $\alpha = 0.05$,

$$z_{\alpha/2}\sqrt{\frac{1}{9n}} = z_{0.025}\sqrt{\frac{1}{9(100)}} \approx 1.960\sqrt{\frac{1}{900}} \approx 0.065,$$

and a joint 0.95 confidence region for the true (Cpl, Cp, Cpu) is given by the line segment in (Cpl, Cp, Cpu)-space,

$$\gamma(1.135,\quad 1.250,\quad 1.365) + (1-\gamma)(1.265,\quad 1.250,\quad 1.235),\quad 0 \le \gamma \le 1. \tag{3.59}$$

This line segment casts a shadow in the (Cpl, Cpu) plane, which we show in Figure 3.1. It is perpendicular to the ray of potentiality and intersects it at the point (1.250, 1.250).

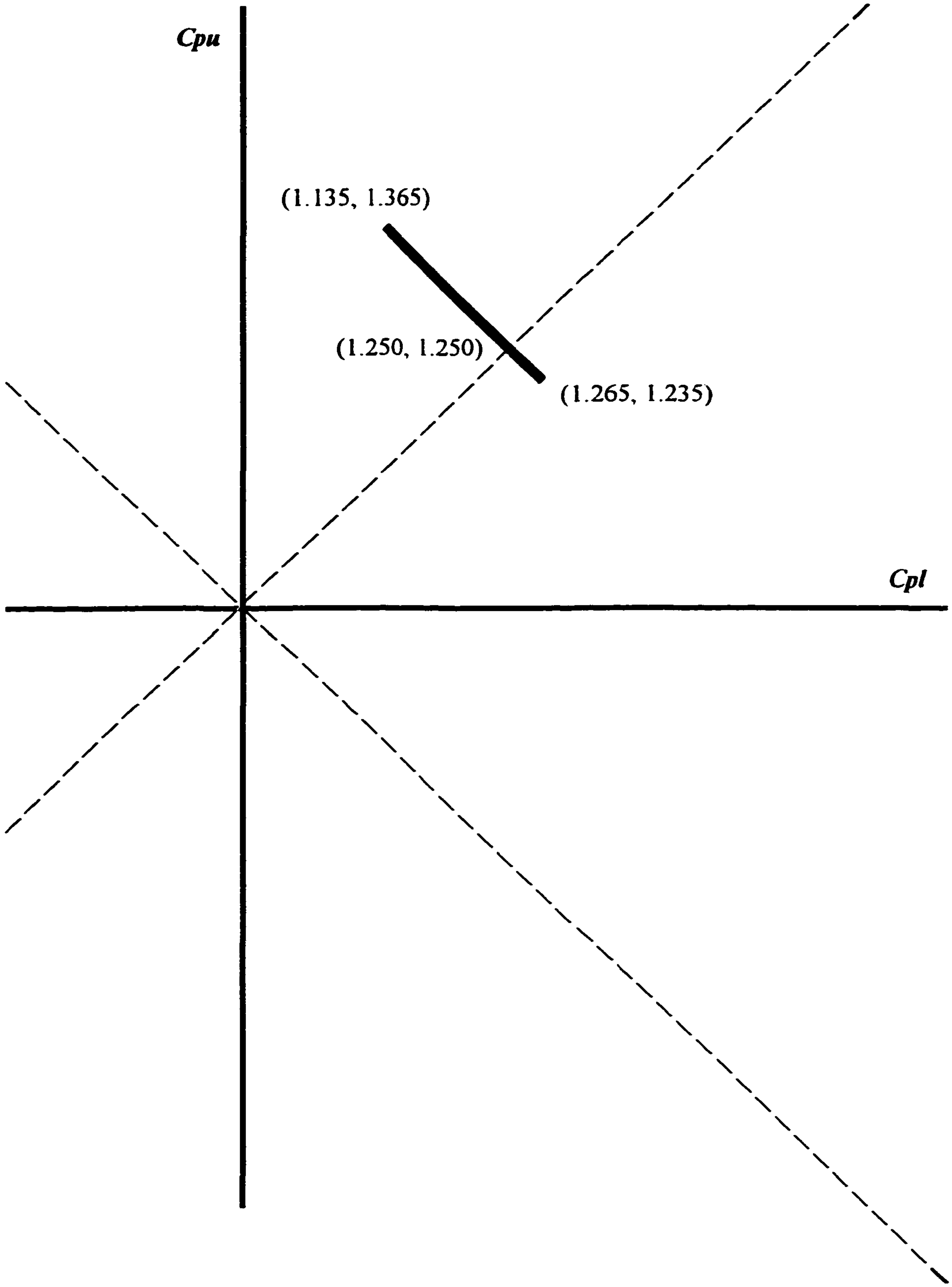

Figure 3.1. 0.95 Joint Confidence Line for True (*Cpl*, *Cpu*), Unknown μ, Known σ

A 0.95 confidence interval for the true proportion π_0 of product outside specification can be gotten. Since

$$\Phi[-3(1.135)] + \Phi[-3(1.365)] = \Phi[-3.405] + \Phi[-4.095]$$

$$= 0.0003309 + 0.0000211 = 0.0003520, \tag{3.60}$$

$$\Phi[-3(1.265)] + \Phi[-3(1.235)] = \Phi[-3.795] + \Phi[-3.705]$$

$$= 0.0000739 + 0.0001057 = 0.0001796, \tag{3.61}$$

$$\Phi[-3(1.250)] + \Phi[-3(1.250)] = \Phi[-3.750] + \Phi[-3.750]$$

$$= 0.0000885 + 0.0000885 = 0.0001770, \tag{3.62}$$

a 0.95 confidence interval for the true proportion π_0 of product outside specification is given by

$$[0.0001770, \quad 0.0003520]. \tag{3.63}$$

Since $\hat{C}p = Cp$ is known, $2\Phi[-3\hat{C}p] = 2\Phi[-3Cp]$ is the potential proportion of product outside specification, if the process could be centered in the specification interval without disturbing the process standard deviation σ. We shall denote it as

$$\pi_{0,\, potential} = 2\Phi[-3Cp]$$

$$= 2\Phi[-3(1.250)] = 2\Phi[-3.750]$$

$$= 2(0.0000885) = 0.0001770. \tag{3.64}$$

Note that this is equal to the lower limit of the true π_0 of equation (3.63), as it should be.

At this point, one may be excused for believing that the (*Cpl*, *Cp*, *Cpu*) parameterization does not lend itself to interval estimation as readily as the (μ, σ) parameterization. But computing the 0.95 confidence interval for the true proportion π_0 of product outside specification directly from the statistic $(\bar{X}, \sigma)$ rather than $(\hat{C}pl, \hat{C}pu)$ takes the same number of calculations.

This case of unknown μ yet known σ may at first appear to be rather artificial. But consider a normal process which has the physical tendency to "slip rigidly." That is, the spread of the process tends to remain constant while the center of the process may change. In such a scenario, management may have a very precise estimate of σ but are less certain about μ.

3.5. Estimating (*Cpl, Cp, Cpu*) when μ is Known and σ is Unknown

Consider the natural estimators

$$(\hat{C}pl, \hat{C}p, \hat{C}pu) = \left(\frac{\mu - LSL}{3S}, \quad \frac{USL - LSL}{6S}, \quad \frac{USL - \mu}{3S} \right), \tag{3.65}$$

where

$$S = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2}\,. \tag{3.66}$$

We have

$$\begin{aligned} E[\hat{C}pl] &= E\left[\frac{\mu - LSL}{3S}\right] = \frac{\mu - LSL}{3\sigma} E\left[\frac{\sigma}{S}\right] = Cpl\; E\left[\frac{\hat{C}p}{Cp}\right] \\ &= Cpl\,\frac{\Gamma[(n-1)/2]}{\Gamma[n/2]}\sqrt{\frac{n}{2}} \end{aligned} \tag{3.67}$$

and

$$\begin{aligned} Var[\hat{C}pl] &= Var\left[\frac{\mu - LSL}{3S}\right] = \left\{\frac{\mu - LSL}{3\sigma}\right\}^2 Var\left[\frac{\sigma}{S}\right] = C^2pl\; Var\left[\frac{\hat{C}p}{Cp}\right] \\ &= C^2pl\left\{\frac{n}{n-2} - E^2\left[\frac{\hat{C}p}{Cp}\right]\right\} \\ &= C^2pl\left\{\frac{n}{n-2} - \left\{\frac{\Gamma[(n-1)/2]}{\Gamma[n/2]}\sqrt{\frac{n}{2}}\right\}^2\right\} \end{aligned}$$

$$= C^2 pl \left\{ \frac{n}{n-2} - \frac{n}{2} \frac{\Gamma^2[(n-1)/2]}{\Gamma^2[n/2]} \right\}. \tag{3.68}$$

We also have

$$E[\hat{C}pu] = E\left[\frac{USL-\mu}{3S}\right] = \frac{USL-\mu}{3\sigma} E\left[\frac{\sigma}{S}\right] = Cpu\, E\left[\frac{\hat{C}p}{Cp}\right]$$

$$= Cpu \frac{\Gamma[(n-1)/2]}{\Gamma[n/2]} \sqrt{\frac{n}{2}} \tag{3.69}$$

and

$$Var[\hat{C}pu] = Var\left[\frac{USL-\mu}{3S}\right] = \left\{\frac{USL-\mu}{3\sigma}\right\}^2 Var\left[\frac{\sigma}{S}\right] = C^2 pu\, Var\left[\frac{\hat{C}p}{Cp}\right]$$

$$= C^2 pu \left\{ \frac{n}{n-2} - E^2\left[\frac{\hat{C}p}{Cp}\right] \right\}$$

$$= C^2 pu \left\{ \frac{n}{n-2} - \left\{ \frac{\Gamma[(n-1)/2]}{\Gamma[n/2]} \sqrt{\frac{n}{2}} \right\}^2 \right\}$$

$$= C^2 pu \left\{ \frac{n}{n-2} - \frac{n}{2} \frac{\Gamma^2[(n-1)/2]}{\Gamma^2[n/2]} \right\}. \tag{3.70}$$

The covariance can be gotten as

$$Cov[\hat{C}pl, \hat{C}pu] = Cov\left[\frac{\mu-LSL}{3S}, \frac{USL-\mu}{3S}\right]$$

$$= \frac{\mu-LSL}{3} \frac{USL-\mu}{3} Cov\left[\frac{1}{S}, \frac{1}{S}\right] = \frac{\mu-LSL}{3} \frac{USL-\mu}{3} Var\left[\frac{1}{S}\right]$$

$$= \frac{\mu-LSL}{3\sigma} \frac{USL-\mu}{3\sigma} Var\left[\frac{\sigma}{S}\right] = Cpl\, Cpu\, Var\left[\frac{\hat{C}p}{Cp}\right]$$

$$= Cpl\, Cpu \left\{ \frac{n}{n-2} - E^2\left[\frac{\hat{C}p}{Cp}\right] \right\}$$

$$= Cpl\, Cpu \left\{ \frac{n}{n-2} - \left\{ \frac{\Gamma[(n-1)/2]}{\Gamma[n/2]} \sqrt{\frac{n}{2}} \right\}^2 \right\}$$

$$= Cpl\, Cpu \left\{ \frac{n}{n-2} - \frac{n}{2} \frac{\Gamma^2[(n-1)/2]}{\Gamma^2[n/2]} \right\}. \tag{3.71}$$

Note that $Cov[\hat{C}pl, \hat{C}pu]$ can be any real number, positive, negative, or zero. Also, we can see that

$$Corr[\hat{C}pl, \hat{C}pu] = \frac{Cov[\hat{C}pl, \hat{C}pu]}{\sqrt{Var[\hat{C}pl]}\sqrt{Var[\hat{C}pu]}}$$

$$= \frac{Cpl\, Cpu \left\{ \frac{n}{n-2} - \frac{n}{2} \frac{\Gamma^2[(n-1)/2]}{\Gamma^2[n/2]} \right\}}{\sqrt{C^2 pl \left\{ \frac{n}{n-2} - \frac{n}{2} \frac{\Gamma^2[(n-1)/2]}{\Gamma^2[n/2]} \right\}}\sqrt{C^2 pu \left\{ \frac{n}{n-2} - \frac{n}{2} \frac{\Gamma^2[(n-1)/2]}{\Gamma^2[n/2]} \right\}}}$$

$$= \frac{Cpl\, Cpu}{|Cpl|\,|Cpu|} = \pm 1, \tag{3.72}$$

provided μ does not fall at either LSL or USL. If μ falls within the specification limits, then $Cov[\hat{C}pl, \hat{C}pu]$ is positive one. If μ falls outside, then it is negative one. There are problems at the two poles $\mu = LSL$ and $\mu = USL$, at which $\hat{C}pl$ or $\hat{C}pu$ is identically zero, and so $Var[\hat{C}pl]$ or $Var[\hat{C}pu]$ is zero. Note further that

$$E[\hat{C}p] = E\left[\frac{1}{2}(\hat{C}pl + \hat{C}pl)\right]$$

$$= \frac{1}{2} Cpl \frac{\Gamma[(n-1)/2]}{\Gamma[n/2]}\sqrt{\frac{n}{2}} + \frac{1}{2} Cpu \frac{\Gamma[(n-1)/2]}{\Gamma[n/2]}\sqrt{\frac{n}{2}}$$

$$= \frac{1}{2}(Cpl + Cpu)\frac{\Gamma[(n-1)/2]}{\Gamma[n/2]}\sqrt{\frac{n}{2}}$$

$$= Cp \frac{\Gamma[(n-1)/2]}{\Gamma[n/2]}\sqrt{\frac{n}{2}} \tag{3.73}$$

and

$$Var\left[\hat{C}p\right] = Var\left[\frac{1}{2}\left(\hat{C}pl + \hat{C}pl\right)\right]$$

$$= \frac{1}{4} Var\left[\hat{C}pl\right] + \frac{1}{4} Var\left[\hat{C}pu\right] + \frac{2}{4} Cov\left[\hat{C}pl, \hat{C}pu\right]$$

$$= \left(\frac{1}{4}C^2pl + \frac{1}{4}C^2pu + \frac{2}{4}Cpl\, Cpu\right)\left\{\frac{n}{n-2} - \left\{\frac{\Gamma[(n-1)/2]}{\Gamma[n/2]}\sqrt{\frac{n}{2}}\right\}^2\right\}$$

$$= \left\{\frac{1}{2}(Cpl + Cpu)\right\}^2\left\{\frac{n}{n-2} - \left\{\frac{\Gamma[(n-1)/2]}{\Gamma[n/2]}\sqrt{\frac{n}{2}}\right\}^2\right\}$$

$$= C^2p\left\{\frac{n}{n-2} - \left\{\frac{\Gamma[(n-1)/2]}{\Gamma[n/2]}\sqrt{\frac{n}{2}}\right\}^2\right\}, \tag{3.74}$$

which are consistent with equations (3.11) and (3.12).

With μ known and σ unknown, the statistic

$$\left(\hat{C}pl, \hat{C}p, \hat{C}pu\right) = \left(\frac{\mu - LSL}{3S}, \quad \frac{USL - LSL}{6S}, \quad \frac{USL - \mu}{3S}\right)$$

has one degree of freedom represented by the random variable

$$S = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2}.$$

It is the maximum likelihood (ML) estimator for the triple index (Cpl, Cp, Cpu), that is,

$$\left(\hat{C}pl_{ML}, \hat{C}p_{ML}, \hat{C}pu_{ML}\right) = \left(\frac{\mu - LSL}{3S}, \quad \frac{USL - LSL}{6S}, \quad \frac{USL - \mu}{3S}\right). \tag{3.75}$$

To get the uniformly minimum variance unbiased (UMVU) estimator for (Cpl, Cp, Cpu), we can take each coordinate of the ML estimator times the "unbiasing factor" to get

$$\left(\hat{C}pl_{UMVU}, \hat{C}p_{UMVU}, \hat{C}pu_{UMVU}\right) = \frac{\Gamma[n/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{2}{n}}\left(\hat{C}pl_{ML}, \hat{C}p_{ML}, \hat{C}pu_{ML}\right)$$

$$= \frac{\Gamma[n/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{2}{n}}\left(\frac{\mu - LSL}{3S}, \quad \frac{USL - LSL}{6S}, \quad \frac{USL - \mu}{3S}\right). \tag{3.76}$$

A $(1 - \alpha)$ confidence interval for the true Cp is given by

$$\left[\hat{C}p\sqrt{\frac{\chi^2_{n,\alpha/2}}{n}},\ \hat{C}p\sqrt{\frac{\chi^2_{n,1-\alpha/2}}{n}}\right], \tag{3.77}$$

where $\chi^2_{n,\alpha/2}$ and $\chi^2_{n,1-\alpha/2}$ are the lower $\alpha/2$ and $(1 - \alpha/2)$ percentiles of the chi-squared random variable with n degrees of freedom. Similarly, a $(1 - \alpha)$ confidence interval for the true Cpl is

$$\left[\hat{C}pl\sqrt{\frac{\chi^2_{n,\alpha/2}}{n}},\ \hat{C}pl\sqrt{\frac{\chi^2_{n,1-\alpha/2}}{n}}\right], \tag{3.78}$$

while a $(1 - \alpha)$ confidence interval for the true Cpu is

$$\left[\hat{C}pu\sqrt{\frac{\chi^2_{n,\alpha/2}}{n}},\ \hat{C}pu\sqrt{\frac{\chi^2_{n,1-\alpha/2}}{n}}\right]. \tag{3.79}$$

A joint $(1 - \alpha)$ confidence region for the true (Cpl, Cpu) is given by the line segment joining the points in the (Cpl, Cpu) plane,

$$\left(\hat{C}pl\sqrt{\frac{\chi^2_{n,\alpha/2}}{n}},\ \hat{C}pu\sqrt{\frac{\chi^2_{n,\alpha/2}}{n}}\right)$$

$$\text{and}\quad \left(\hat{C}pl\sqrt{\frac{\chi^2_{n,1-\alpha/2}}{n}},\ \hat{C}pu\sqrt{\frac{\chi^2_{n,1-\alpha/2}}{n}}\right). \tag{3.80}$$

In other words, a joint $(1 - \alpha)$ confidence region for the true (Cpl, Cpu) is given by the line segment in the (Cpl, Cpu) plane,

$$\gamma\left(\hat{C}pl\sqrt{\frac{\chi^2_{n,\alpha/2}}{n}},\ \hat{C}pu\sqrt{\frac{\chi^2_{n,\alpha/2}}{n}}\right)$$

$$+(1-\gamma)\left(\hat{C}pl\sqrt{\frac{\chi^2_{n,1-\alpha/2}}{n}},\quad \hat{C}pu\sqrt{\frac{\chi^2_{n,1-\alpha/2}}{n}}\right) \tag{3.81}$$

for all real $0 \le \gamma \le 1$. Also, a joint $(1 - \alpha)$ confidence region for the true (Cpl, Cp, Cpu) is given by the line segment in (Cpl, Cp, Cpu)-space,

$$\gamma\left(\hat{C}pl\sqrt{\frac{\chi^2_{n,\alpha/2}}{n}},\quad \hat{C}p\sqrt{\frac{\chi^2_{n,\alpha/2}}{n}},\quad \hat{C}pu\sqrt{\frac{\chi^2_{n,\alpha/2}}{n}}\right)$$

$$+(1-\gamma)\left(\hat{C}pl\sqrt{\frac{\chi^2_{n,1-\alpha/2}}{n}},\quad \hat{C}p\sqrt{\frac{\chi^2_{n,1-\alpha/2}}{n}},\quad \hat{C}pu\sqrt{\frac{\chi^2_{n,1-\alpha/2}}{n}}\right) \tag{3.82}$$

for all real $0 \le \gamma \le 1$.

To take a numerical example, suppose we have estimated

$$\left(\hat{C}pl,\quad \hat{C}p,\quad \hat{C}pu\right) = (1.200,\quad 1.250,\quad 1.300) \tag{3.83}$$

based on $n = 100$. For $\alpha = 0.05$,

$$\sqrt{\frac{\chi^2_{n,1-\alpha/2}}{n}} = \sqrt{\frac{\chi^2_{100,0.975}}{100}} \approx \sqrt{\frac{129.561}{100}} \approx 1.138248655,$$

$$\sqrt{\frac{\chi^2_{n,\alpha/2}}{n}} = \sqrt{\frac{\chi^2_{100,0.025}}{100}} \approx \sqrt{\frac{74.2219}{100}} \approx 0.861521329,$$

and a joint 0.95 confidence region for the true (Cpl, Cp, Cpu) is given by the line segment in (Cpl, Cp, Cpu)-space,

$$\gamma(1.034,\quad 1.077,\quad 1.120)+(1-\gamma)(1.366,\quad 1.423,\quad 1.480),\quad 0 \le \gamma \le 1 \,. \tag{3.84}$$

On the (Cpl, Cp, Cpu) diagram of Figure 3.2, this line segment can be imagined as two line segments. The first line segment lies in the subspace spanned by $\left(\hat{C}pl, \hat{C}pu\right) = (1.200, 1.300)$ and runs from the point (1.034, 1.120) to the point (1.366, 1.480). It represents a 0.95 joint confidence region or line for the true (Cpl, Cpu). The second line segment lies on the ray of

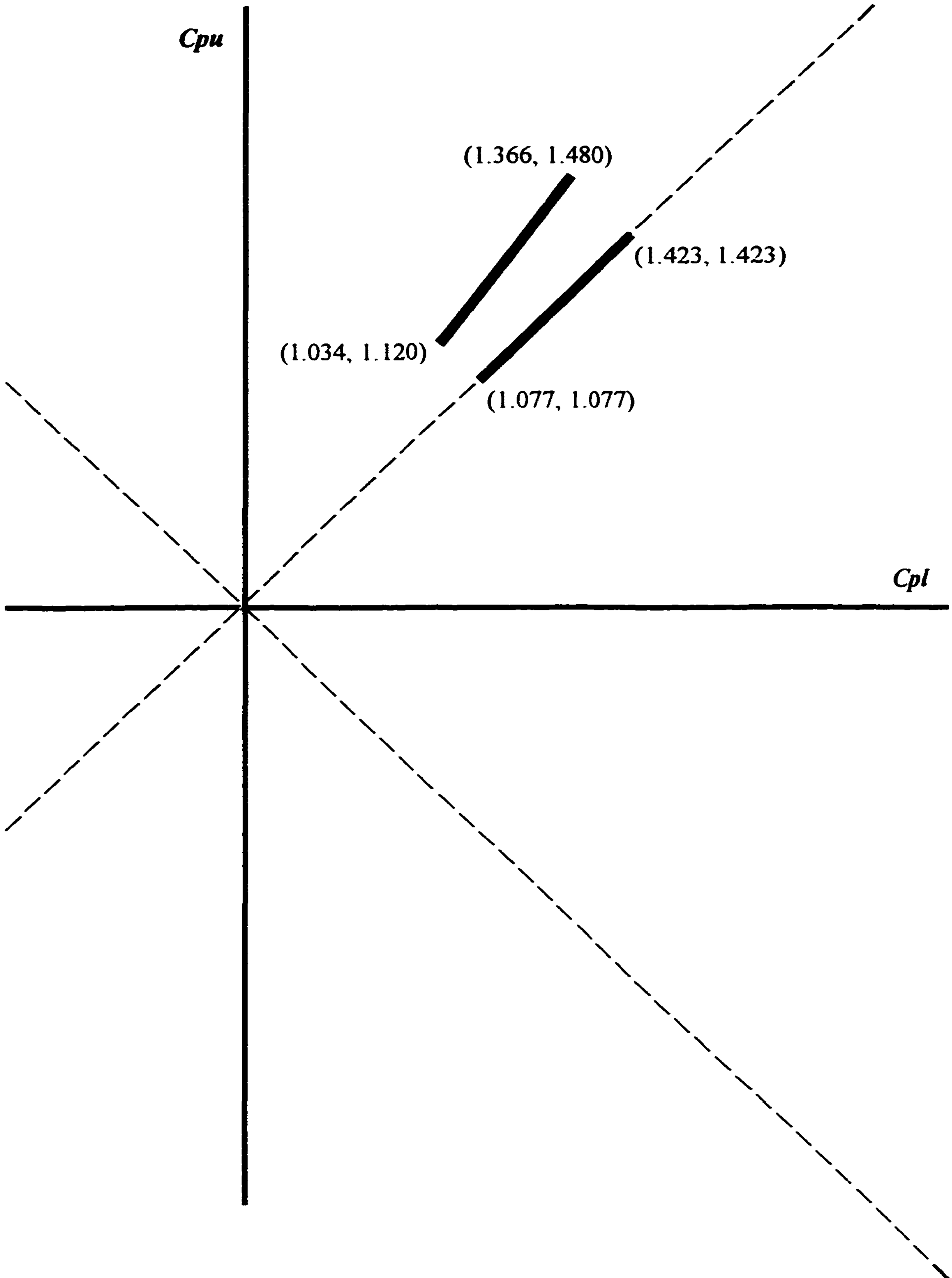

Figure 3.2. 0.95 Joint Confidence Line for True (*Cpl*, *Cpu*), Known μ, Unknown σ

potentiality and runs from the point (1.077, 1.077) to the point (1.423, 1.423). It represents a 0.95 confidence interval for the true *Cp*. Since the estimator of (*Cpl*, *Cp*, *Cpu*) has only one degree of freedom, the total joint confidence is 0.95. Of course, if μ lies at the midpoint *m* of the specification interval, then the two line segments coincide.

A 0.95 confidence interval for the true proportion π_0 of product outside specification can be gotten. Since

$$\Phi[-3(1.034)] + \Phi[-3(1.120)] = \Phi[-3.102] + \Phi[-3.360]$$
$$= 0.0009612 + 0.0003898 = 0.0013510 \tag{3.85}$$

and

$$\Phi[-3(1.366)] + \Phi[-3(1.480)] = \Phi[-4.098] + \Phi[-4.440]$$
$$= 0.0000209 + 0.0000045 = 0.0000254, \tag{3.86}$$

a 0.95 confidence interval for the true proportion π_0 of product outside specification is given by

$$[0.0000254, \quad 0.0013510]. \tag{3.87}$$

From the *Cp* confidence interval, we can get a confidence interval for a *potential* π_0. Since

$$2\Phi[-3(1.077)] = 2\Phi[-3.231] = 2(0.0006168) = 0.0012336, \tag{3.88}$$

$$2\Phi[-3(1.423)] = 2\Phi[-4.269] = 2(0.0000098) = 0.0000196, \tag{3.89}$$

we have

$$[0.0000196, \quad 0.0012336]. \tag{3.90}$$

Observe that the lower limit of this 0.95 confidence interval for potential π_0 falls below the lower limit of the 0.95 confidence interval for the current π_0, as it should. Also, the upper limit of this 0.95 confidence interval for potential π_0 falls below the upper limit of the 0.95 confidence interval for the current π_0, as it should.

Again, one may begin to suspect it better to make the computations with the realization of S. But computing the 0.95 confidence interval for the true proportion π_0 of product outside specification directly from the statistic (μ, S) rather than $(\hat{Cpl}, \hat{Cpu})$ takes the same number of calculations. There is no advantage in returning to (μ, S) for these calculations, once given $(\hat{Cpl}, \hat{Cpu})$. The equivalence is undeniable.

This case of known μ but unknown σ may at first appear to be rather artificial. We may imagine a normal process which becomes diffuse over time while maintaining its center. In this situation, management may have a very precise estimate of μ but are less certain about the spread of the process as measured by σ.

3.6. Estimating (*Cpl*, *Cp*, *Cpu*) when Both μ and σ are Unknown

Consider the natural estimators

$$(\hat{Cpl}, \hat{Cp}, \hat{Cpu}) = \left(\frac{\bar{X} - LSL}{3S}, \frac{USL - LSL}{6S}, \frac{USL - \bar{X}}{3S} \right), \tag{3.91}$$

where

$$(\bar{X}, S) = \left(\frac{1}{n} \sum_{i=1}^{n} X_i, \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2} \right). \tag{3.92}$$

Again we see an identity,

$$\hat{Cp} = \frac{1}{2}(\hat{Cpl} + \hat{Cpu}). \tag{3.93}$$

We will need the definition of the noncentral t random variable. Consider the random variable $t'_{\nu,\delta} = \dfrac{Z + \delta}{\sqrt{\chi^2_\nu / \nu}}$ where Z is standard normal, χ^2_ν is chi-squared with ν degrees of freedom, δ is a real constant, and Z and χ_ν^2 are statistically independent. The random variable $t'_{\nu,\delta}$ is called the noncentral t with degrees of freedom ν and noncentrality parameter δ. On writing

$$t'_{\nu,\delta} = \frac{Z+\delta}{\sqrt{\chi_\nu^2/\nu}} = \frac{Z}{\sqrt{\chi_\nu^2/\nu}} + \frac{\delta}{\sqrt{\chi_\nu^2/\nu}} = t_\nu + \delta\sqrt{\nu}\,\chi_\nu^{-1}\ . \tag{3.94}$$

we see that the noncentral t is the sum of a central t and a multiple of an inverted chi, each statistically independent of the other. Deriving the mean and variance,

$$\begin{aligned} E\left[t'_{\nu,\delta}\right] &= E\left[t_\nu + \delta\sqrt{\nu}\,\chi_\nu^{-1}\right] \\ &= \delta\sqrt{\nu}E\left[\chi_\nu^{-1}\right] = \delta\sqrt{\nu}\frac{\Gamma[(\nu-1)/2]}{\Gamma[\nu/2]}\frac{1}{\sqrt{2}} \\ &= \delta\frac{\Gamma[(\nu-1)/2]}{\Gamma[\nu/2]}\sqrt{\frac{\nu}{2}} \end{aligned} \tag{3.95}$$

and

$$\begin{aligned} Var\left[t'_{\nu,\delta}\right] &= Var\left[t_\nu + \delta\sqrt{\nu}\,\chi_\nu^{-1}\right] = Var\left[t_\nu\right] + \delta^2\nu Var\left[\chi_\nu^{-1}\right] \\ &= \frac{\nu}{\nu-2} + \delta^2\nu\left\{\frac{1}{\nu-2} - \left\{\frac{\Gamma[(\nu-1)/2]}{\Gamma[\nu/2]}\frac{1}{\sqrt{2}}\right\}^2\right\} \\ &= \frac{\nu}{\nu-2}\left(1+\delta^2\right) - \left\{\delta\frac{\Gamma[(\nu-1)/2]}{\Gamma[\nu/2]}\sqrt{\frac{\nu}{2}}\right\}^2 \\ &= \frac{\nu}{\nu-2}\left(1+\delta^2\right) - E^2\left[t'_{\nu,\delta}\right]. \end{aligned} \tag{3.96}$$

Now consider the natural estimator of Cpl given by

$$\hat{C}pl = \frac{\overline{X}-LSL}{3S} = \frac{\overline{X}-\mu+\mu-LSL}{3S} = \frac{\dfrac{\overline{X}-\mu}{\sigma/\sqrt{n}} + \dfrac{\mu-LSL}{\sigma/\sqrt{n}}}{\dfrac{3S}{\sigma/\sqrt{n}}}$$

$$= \frac{1}{3\sqrt{n}}\left[\frac{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}+\frac{\mu-LSL}{\sigma/\sqrt{n}}}{\sqrt{\frac{S^2}{\sigma^2}}}\right] = \frac{1}{3\sqrt{n}}\left[\frac{Z+\delta_l}{\sqrt{\chi^2_{n-1}/(n-1)}}\right] = \frac{t'_{n-1,\delta_l}}{3\sqrt{n}}, \tag{3.97}$$

where

$$\delta_l = \frac{\mu-LSL}{\sigma/\sqrt{n}} = 3\sqrt{n}\,\frac{\mu-LSL}{3\sigma} = 3\sqrt{n}\,Cpl. \tag{3.98}$$

This yields

$$\begin{aligned} E[\hat{C}pl] &= E\left[\frac{t'_{n-1,\delta_l}}{3\sqrt{n}}\right] = \frac{1}{3\sqrt{n}}\delta_l\frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}} \\ &= Cpl\frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}} = Cpl\,E\left[\frac{\hat{C}p}{Cp}\right] \end{aligned} \tag{3.99}$$

and

$$\begin{aligned} Var[\hat{C}pl] &= Var\left[\frac{t'_{n-1,\delta_l}}{3\sqrt{n}}\right] = \frac{1}{9n}Var\left[t'_{n-1,\delta_l}\right] \\ &= \frac{1}{9n}\left\{\frac{n-1}{n-3}\left(1+\delta_l^2\right)-\left\{\delta_l\frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}}\right\}^2\right\} \\ &= \frac{1}{9n}\frac{n-1}{n-3}\left(1+\delta_l^2\right)-\left\{\frac{\delta_l}{3\sqrt{n}}\frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}}\right\}^2 \\ &= \frac{1}{9n}\frac{n-1}{n-3}\left(1+\delta_l^2\right)-E^2[\hat{C}pl]. \\ &= \frac{1}{9n}\frac{n-1}{n-3}+\frac{1}{9n}\frac{n-1}{n-3}9n\,C^2pl-C^2pl\,E^2\left[\frac{\hat{C}p}{Cp}\right] \\ &= \frac{1}{9n}\frac{n-1}{n-3}+C^2pl\left\{\frac{n-1}{n-3}-E^2\left[\frac{\hat{C}p}{Cp}\right]\right\} \\ &= \frac{1}{9n}\frac{n-1}{n-3}+C^2pl\,Var\left[\frac{\hat{C}p}{Cp}\right] \end{aligned}$$

$$= \frac{1}{9n}\frac{n-1}{n-3} + C^2 pl \left\{ \frac{n-1}{n-3} - \left\{ \frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}} \right\}^2 \right\}. \tag{3.100}$$

Next consider the natural estimator of *Cpu* given by

$$\hat{C}pu = \frac{USL - \bar{X}}{3S}. \tag{3.101}$$

By symmetry, we have

$$E[\hat{C}pu] = E\left[\frac{t'_{n-1,\delta_u}}{3\sqrt{n}}\right] = \frac{1}{3\sqrt{n}}\delta_u \frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}}$$

$$= Cpu \frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}} = Cpu\, E\left[\frac{\hat{C}p}{Cp}\right]. \tag{3.102}$$

and

$$Var[\hat{C}pu] = Var\left[\frac{t'_{n-1,\delta_u}}{3\sqrt{n}}\right] = \frac{1}{9n} Var\left[t'_{n-1,\delta_u}\right]$$

$$= \frac{1}{9n}\left\{ \frac{n-1}{n-3}\left(1+\delta_u^2\right) - \left\{ \delta_u \frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}} \right\}^2 \right\}$$

$$= \frac{1}{9n}\frac{n-1}{n-3}\left(1+\delta_u^2\right) - \left\{ \frac{\delta_u}{3\sqrt{n}} \frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}} \right\}^2$$

$$= \frac{1}{9n}\frac{n-1}{n-3}\left(1+\delta_u^2\right) - E^2[\hat{C}pu]$$

$$= \frac{1}{9n}\frac{n-1}{n-3} + \frac{1}{9n}\frac{n-1}{n-3} 9n\, C^2 pu - C^2 pu\, E^2\left[\frac{\hat{C}p}{Cp}\right]$$

$$= \frac{1}{9n}\frac{n-1}{n-3} + C^2 pu \left\{ \frac{n-1}{n-3} - E^2\left[\frac{\hat{C}p}{Cp}\right] \right\}$$

$$= \frac{1}{9n}\frac{n-1}{n-3} + C^2 pu\, Var\left[\frac{\hat{C}p}{Cp}\right]$$

$$= \frac{1}{9n}\frac{n-1}{n-3} + C^2 pu \left\{ \frac{n-1}{n-3} - \left\{ \frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}} \right\}^2 \right\}, \tag{3.103}$$

where

$$\delta_u = \frac{USL-\mu}{\sigma/\sqrt{n}} = 3\sqrt{n}\,\frac{USL-\mu}{3\sigma} = 3\sqrt{n}\,Cpu. \tag{3.104}$$

The covariance can be gotten as

$$\begin{aligned}
Cov\left[\hat{C}pl, \hat{C}pu\right] &= Cov\left[\frac{\bar{X}-LSL}{3S}, \frac{USL-\bar{X}}{3S}\right] \\
&= E\left[\frac{\bar{X}-LSL}{3S}\frac{USL-\bar{X}}{3S}\right] - E\left[\frac{\bar{X}-LSL}{3S}\right]E\left[\frac{USL-\bar{X}}{3S}\right] \\
&= E\left[\frac{1}{9S^2}\right]E\left[(\bar{X}-LSL)(USL-\bar{X})\right] - Cpl\,E\left[\frac{\hat{C}p}{Cp}\right]Cpu\,E\left[\frac{\hat{C}p}{Cp}\right] \\
&= E\left[\frac{1}{9S^2}\right]\left\{USL\,E\left[\bar{X}\right] - E\left[\bar{X}^2\right] - LSL\,USL + LSL\,E\left[\bar{X}\right]\right\} - Cpl\,Cpu\,E^2\left[\frac{\hat{C}p}{Cp}\right] \\
&= E\left[\frac{1}{9S^2}\right]\left\{USL\,\mu - \left\{\frac{\sigma^2}{n} + \mu^2\right\} - LSL\,USL + LSL\,\mu\right\} - Cpl\,Cpu\,E^2\left[\frac{\hat{C}p}{Cp}\right] \\
&= E\left[\frac{1}{9S^2}\right]\left\{(\mu - LSL)(USL-\mu) - \frac{\sigma^2}{n}\right\} - Cpl\,Cpu\,E^2\left[\frac{\hat{C}p}{Cp}\right] \\
&= E\left[\frac{\sigma^2}{S^2}\right]\left\{\left(\frac{\mu-LSL}{3\sigma}\right)\left(\frac{USL-\mu}{3\sigma}\right) - \frac{1}{9n}\right\} - Cpl\,Cpu\,E^2\left[\frac{\hat{C}p}{Cp}\right] \\
&= \left\{Var\left[\frac{\hat{C}p}{Cp}\right] + E^2\left[\frac{\hat{C}p}{Cp}\right]\right\}\left\{Cpl\,Cpu - \frac{1}{9n}\right\} - Cpl\,Cpu\,E^2\left[\frac{\hat{C}p}{Cp}\right] \\
&= Var\left[\frac{\hat{C}p}{Cp}\right]\left\{Cpl\,Cpu - \frac{1}{9n}\right\} - \frac{1}{9n}E^2\left[\frac{\hat{C}p}{Cp}\right] \\
&= Cpl\,Cpu\,Var\left[\frac{\hat{C}p}{Cp}\right] - \frac{1}{9n}E\left[\left(\frac{\hat{C}p}{Cp}\right)^2\right] \\
&= -\frac{1}{9n}\frac{n-1}{n-3} + Cpl\,Cpu\,Var\left[\frac{\hat{C}p}{Cp}\right] \\
&= -\frac{1}{9n}\frac{n-1}{n-3} + Cpl\,Cpu\left\{\frac{n-1}{n-3} - \left\{\frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}}\right\}^2\right\}.
\end{aligned} \tag{3.105}$$

Note further that

$$E\left[\hat{C}p\right] = E\left[\frac{1}{2}\left(\hat{C}pl + \hat{C}pl\right)\right]$$

$$= \frac{1}{2}Cpl\,\frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}} + \frac{1}{2}Cpu\,\frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}}$$

$$= \frac{1}{2}(Cpl + Cpu)\,\frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}}$$

$$= Cp\,\frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}} \qquad (3.106)$$

and

$$Var\left[\hat{C}p\right] = Var\left[\frac{1}{2}\left(\hat{C}pl + \hat{C}pl\right)\right]$$

$$= \frac{1}{4}Var\left[\hat{C}pl\right] + \frac{1}{4}Var\left[\hat{C}pu\right] + \frac{2}{4}Cov\left[\hat{C}pl, \hat{C}pu\right]$$

$$= \frac{1}{4}C^2pl\,Var\left[\frac{\hat{C}p}{Cp}\right] + \frac{1}{4}C^2pu\,Var\left[\frac{\hat{C}p}{Cp}\right] + \frac{2}{4}Cpl\,Cpu\,Var\left[\frac{\hat{C}p}{Cp}\right]$$

$$= \left\{\frac{1}{4}C^2pl + \frac{1}{4}C^2pu + \frac{2}{4}Cpl\,Cpu\right\}Var\left[\frac{\hat{C}p}{Cp}\right]$$

$$= \left\{\frac{1}{2}(Cpl + Cpu)\right\}^2 Var\left[\frac{\hat{C}p}{Cp}\right] = C^2p\,Var\left[\frac{\hat{C}p}{Cp}\right]$$

$$= C^2p\left\{\frac{n-1}{n-3} - \left\{\frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}}\right\}^2\right\}, \qquad (3.107)$$

which are consistent with equations (3.34) and (3.35).

In Chapter 1, we gave the maximum likelihood (ML) and uniformly minimum variance unbiased (UMVU) point estimators for the triple index (*Cpl*, *Cp*, *Cpu*) for the case of independent normal random variates where both μ and σ are unknown. It remains to discuss interval estimation of (*Cpl*, *Cp*, *Cpu*). We have previously demonstrated the equivalence of this task to that of estimating unknown (μ, σ). Of course, in so far as the joint interval estimation of (μ, σ) is not elementary (it is usually omitted in elementary statistics courses), we can expect a similar level of effort ahead of us in our (Cpl, Cp, Cpu) endeavor.

We begin by constructing a $(1 - \alpha)^2$ joint confidence region for (μ, σ) for the case of independent normal random variates. Now it is known that a $(1 - \alpha)$ confidence interval for unknown σ is given by

$$\left[S\sqrt{\frac{n-1}{\chi^2_{n-1,\,1-\alpha/2}}},\quad S\sqrt{\frac{n-1}{\chi^2_{n-1,\,\alpha/2}}} \right]. \tag{3.108}$$

We also know that a $(1 - \alpha)$ confidence interval for unknown μ is given by

$$\left[\overline{X} - t_{n-1,\,\alpha/2}\frac{S}{\sqrt{n}},\quad \overline{X} + t_{n-1,\,\alpha/2}\frac{S}{\sqrt{n}} \right]. \tag{3.109}$$

Yet it will not do to take the Cartesian product of these two confidence intervals and call the result a $(1 - \alpha)^2$ joint confidence region for the true (μ, σ). This is because, even though the random variables $\overline{X}$ and S are independent, clearly $\overline{X} + t_{n-1,\,\alpha/2}\frac{S}{\sqrt{n}}$ and S are not. We must take a different tack.

Observe that if we knew σ, then a $(1 - \alpha)$ confidence interval for the unknown μ would be given by

$$\left[\overline{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}},\quad \overline{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \right]. \tag{3.110}$$

Take the lower endpoint from the σ confidence interval of equation (3.108) and substitute for σ in the μ confidence interval of equation (3.110). Repeat for each element of the σ confidence interval. In this way, we distribute the total $(1 - \alpha)$ confidence of the σ confidence interval across the μ axis. The procedure results in the trapezoidal region depicted in Figure 3.3,

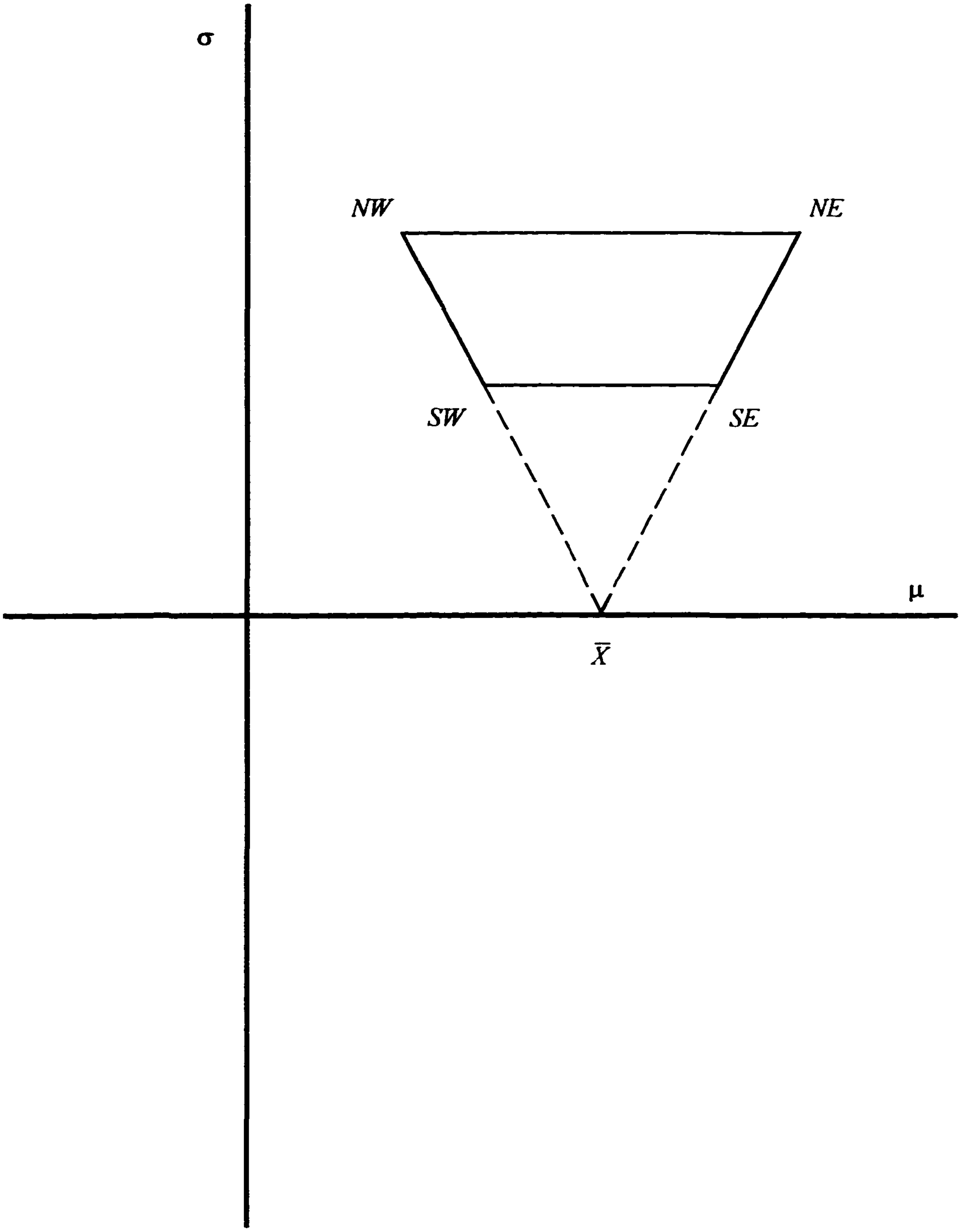

Figure 3.3. $(1 - \alpha)^2$ Joint Confidence Region for True (μ, σ)

representing a $(1 - \alpha)^2$ joint confidence region for the true (μ, σ). For more discussion, the reader is referred to Lindgren (1976). The four vertices of the trapezoid are the points

$$SW \Rightarrow \left(\overline{X} - \frac{z_{\alpha/2}}{\sqrt{n}} S\sqrt{\frac{n-1}{\chi^2_{n-1,1-\alpha/2}}},\quad S\sqrt{\frac{n-1}{\chi^2_{n-1,1-\alpha/2}}} \right), \tag{3.111}$$

$$SE \Rightarrow \left(\overline{X} + \frac{z_{\alpha/2}}{\sqrt{n}} S\sqrt{\frac{n-1}{\chi^2_{n-1,1-\alpha/2}}},\quad S\sqrt{\frac{n-1}{\chi^2_{n-1,1-\alpha/2}}} \right), \tag{3.112}$$

$$NW \Rightarrow \left(\overline{X} - \frac{z_{\alpha/2}}{\sqrt{n}} S\sqrt{\frac{n-1}{\chi^2_{n-1,\alpha/2}}},\quad S\sqrt{\frac{n-1}{\chi^2_{n-1,\alpha/2}}} \right), \tag{3.113}$$

$$NE \Rightarrow \left(\overline{X} + \frac{z_{\alpha/2}}{\sqrt{n}} S\sqrt{\frac{n-1}{\chi^2_{n-1,\alpha/2}}},\quad S\sqrt{\frac{n-1}{\chi^2_{n-1,\alpha/2}}} \right), \tag{3.114}$$

which we indicate in Figure 3.3 by their compass designations. The joint confidence region for the unknown (μ, σ) is the set of all convex combinations of the vertices of the trapezoid given in equations (3.111) through (3.114), that is, the set of points in the (μ, σ) plane,

$$\begin{aligned}
&\alpha \left(\overline{X} - \frac{z_{\alpha/2}}{\sqrt{n}} S\sqrt{\frac{n-1}{\chi^2_{n-1,1-\alpha/2}}},\quad S\sqrt{\frac{n-1}{\chi^2_{n-1,1-\alpha/2}}} \right) \\
&+\beta \left(\overline{X} + \frac{z_{\alpha/2}}{\sqrt{n}} S\sqrt{\frac{n-1}{\chi^2_{n-1,1-\alpha/2}}},\quad S\sqrt{\frac{n-1}{\chi^2_{n-1,1-\alpha/2}}} \right) \\
&+\gamma \left(\overline{X} - \frac{z_{\alpha/2}}{\sqrt{n}} S\sqrt{\frac{n-1}{\chi^2_{n-1,\alpha/2}}},\quad S\sqrt{\frac{n-1}{\chi^2_{n-1,\alpha/2}}} \right) \\
&+\delta \left(\overline{X} + \frac{z_{\alpha/2}}{\sqrt{n}} S\sqrt{\frac{n-1}{\chi^2_{n-1,\alpha/2}}},\quad S\sqrt{\frac{n-1}{\chi^2_{n-1,\alpha/2}}} \right)
\end{aligned} \tag{3.115}$$

for all real α, β, γ, and δ in the interval [0, 1] such that $\alpha + \beta + \gamma + \delta = 1$. In repeated samples, $(1 - \alpha)^2$ of the constructed trapezoids would contain the true (μ, σ). We also note that the trapezoid corresponds to a region of Bayesian posterior probability $(1 - \alpha)^2$ when the prior density of (μ, σ) is taken to be the standard noninformative prior, in which σ is proportional to $1/\sigma$ on the positive reals and independently, μ is proportional to a constant over the reals.

We next demonstrate the construction of a $(1 - \alpha)^2$ joint confidence region for the true (Cpl, Cpu). Given the specification limits LSL and USL, we have

$$(Cpl, Cpu) = \left(\frac{\mu - LSL}{3\sigma}, \ \frac{USL - \mu}{3\sigma} \right). \tag{3.116}$$

We substitute, one at a time, the four vertices of the (μ, σ) confidence region, given by equations (3.111) through (3.114), into equation (3.116) to get the four vertices of a $(1 - \alpha)^2$ joint confidence region for the true (Cpl, Cpu) shown in Figure 3.4,

$$\left(\frac{\bar{X} - \frac{z_{\alpha/2}}{\sqrt{n}} S \sqrt{\frac{n-1}{\chi^2_{n-1,1-\alpha/2}}} - LSL}{3S\sqrt{\frac{n-1}{\chi^2_{n-1,1-\alpha/2}}}}, \ \frac{USL - \bar{X} + \frac{z_{\alpha/2}}{\sqrt{n}} S \sqrt{\frac{n-1}{\chi^2_{n-1,1-\alpha/2}}}}{3S\sqrt{\frac{n-1}{\chi^2_{n-1,1-\alpha/2}}}} \right), \tag{3.117}$$

$$\left(\frac{\bar{X} + \frac{z_{\alpha/2}}{\sqrt{n}} S \sqrt{\frac{n-1}{\chi^2_{n-1,1-\alpha/2}}} - LSL}{3S\sqrt{\frac{n-1}{\chi^2_{n-1,1-\alpha/2}}}}, \ \frac{USL - \bar{X} - \frac{z_{\alpha/2}}{\sqrt{n}} S \sqrt{\frac{n-1}{\chi^2_{n-1,1-\alpha/2}}}}{3S\sqrt{\frac{n-1}{\chi^2_{n-1,1-\alpha/2}}}} \right), \tag{3.118}$$

$$\left(\frac{\bar{X} - \frac{z_{\alpha/2}}{\sqrt{n}} S \sqrt{\frac{n-1}{\chi^2_{n-1,\alpha/2}}} - LSL}{3S\sqrt{\frac{n-1}{\chi^2_{n-1,\alpha/2}}}}, \ \frac{USL - \bar{X} + \frac{z_{\alpha/2}}{\sqrt{n}} S \sqrt{\frac{n-1}{\chi^2_{n-1,\alpha/2}}}}{3S\sqrt{\frac{n-1}{\chi^2_{n-1,\alpha/2}}}} \right), \tag{3.119}$$

$$\left(\frac{\bar{X} + \frac{z_{\alpha/2}}{\sqrt{n}} S \sqrt{\frac{n-1}{\chi^2_{n-1,\alpha/2}}} - LSL}{3S\sqrt{\frac{n-1}{\chi^2_{n-1,\alpha/2}}}}, \ \frac{USL - \bar{X} - \frac{z_{\alpha/2}}{\sqrt{n}} S \sqrt{\frac{n-1}{\chi^2_{n-1,\alpha/2}}}}{3S\sqrt{\frac{n-1}{\chi^2_{n-1,\alpha/2}}}} \right). \tag{3.120}$$

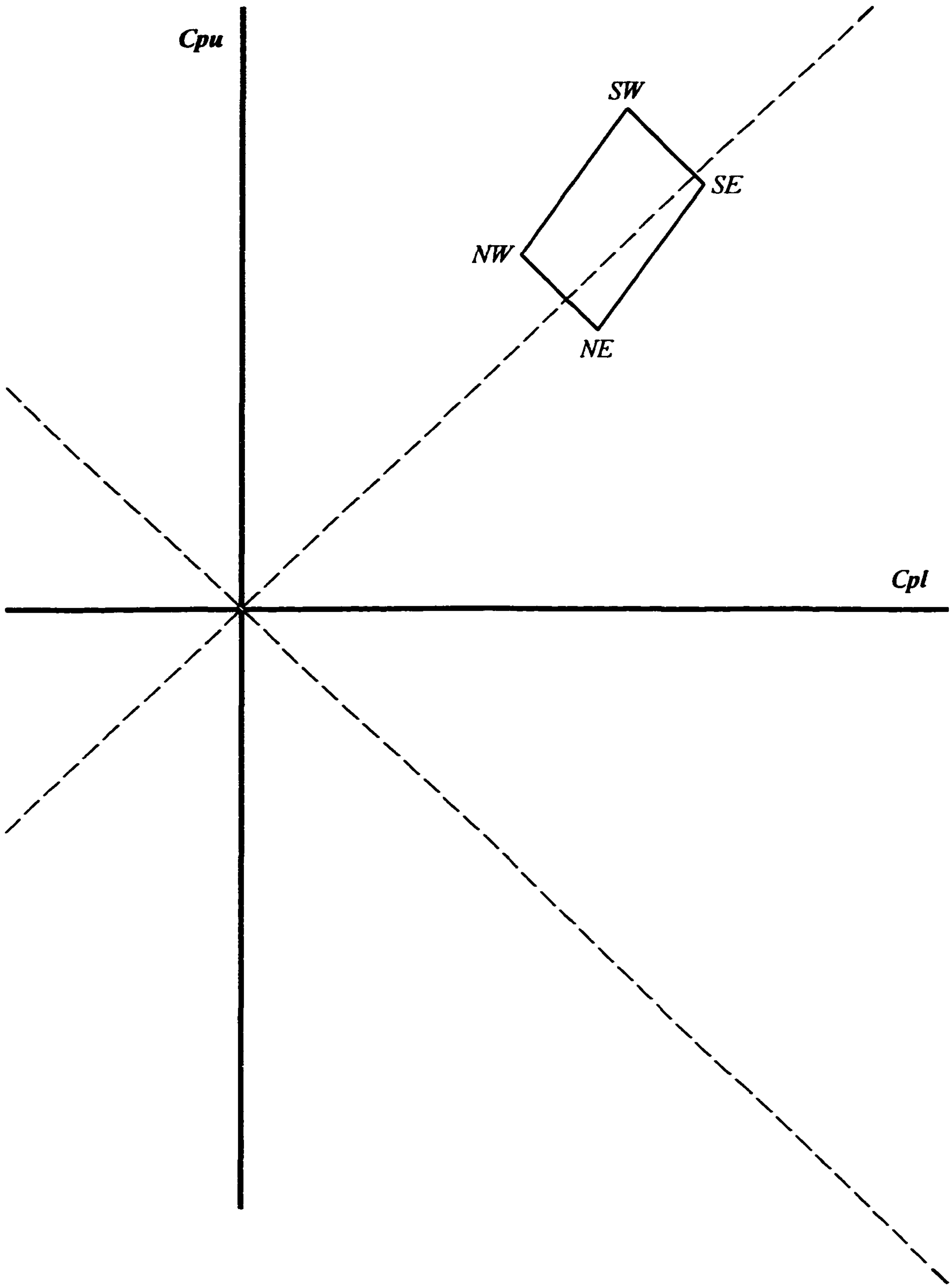

Figure 3.4. $(1 - \alpha)^2$ Joint Confidence Region for True (*Cpl*, *Cpu*), Unknown (μ, σ)

Simplifying yields the four vertices

$$SW \Rightarrow \left(\hat{C}pl\sqrt{\frac{\chi^2_{n-1,1-\alpha/2}}{n-1}} - \frac{z_{\alpha/2}}{3\sqrt{n}},\ \hat{C}pu\sqrt{\frac{\chi^2_{n-1,1-\alpha/2}}{n-1}} + \frac{z_{\alpha/2}}{3\sqrt{n}} \right), \tag{3.121}$$

$$SE \Rightarrow \left(\hat{C}pl\sqrt{\frac{\chi^2_{n-1,1-\alpha/2}}{n-1}} + \frac{z_{\alpha/2}}{3\sqrt{n}},\ \hat{C}pu\sqrt{\frac{\chi^2_{n-1,1-\alpha/2}}{n-1}} - \frac{z_{\alpha/2}}{3\sqrt{n}} \right), \tag{3.122}$$

$$NW \Rightarrow \left(\hat{C}pl\sqrt{\frac{\chi^2_{n-1,\alpha/2}}{n-1}} - \frac{z_{\alpha/2}}{3\sqrt{n}},\ \hat{C}pu\sqrt{\frac{\chi^2_{n-1,\alpha/2}}{n-1}} + \frac{z_{\alpha/2}}{3\sqrt{n}} \right), \tag{3.123}$$

$$NE \Rightarrow \left(\hat{C}pl\sqrt{\frac{\chi^2_{n-1,\alpha/2}}{n-1}} + \frac{z_{\alpha/2}}{3\sqrt{n}},\ \hat{C}pu\sqrt{\frac{\chi^2_{n-1,\alpha/2}}{n-1}} - \frac{z_{\alpha/2}}{3\sqrt{n}} \right), \tag{3.124}$$

where we have kept the compass labels from the (μ, σ) confidence region. Note this carefully, in as much as the "northern" points now appear below the "southern" points in the (Cpl, Cpu) confidence region of Figure 3.4.

To see that the set of all convex combinations of these four vertices form a parallelogram in the (Cpl, Cpu) plane, rewrite the vertices as

$$SW \Rightarrow \sqrt{\frac{\chi^2_{n-1,1-\alpha/2}}{n-1}}\left(\hat{C}pl,\ \hat{C}pu\right) + \frac{z_{\alpha/2}}{\sqrt{9n}}(-1,\ 1), \tag{3.125}$$

$$SE \Rightarrow \sqrt{\frac{\chi^2_{n-1,1-\alpha/2}}{n-1}}\left(\hat{C}pl,\ \hat{C}pu\right) + \frac{z_{\alpha/2}}{\sqrt{9n}}(1,\ -1), \tag{3.126}$$

$$NW \Rightarrow \sqrt{\frac{\chi^2_{n-1,\alpha/2}}{n-1}}\left(\hat{C}pl,\ \hat{C}pu\right) + \frac{z_{\alpha/2}}{\sqrt{9n}}(-1,\ 1), \tag{3.127}$$

$$NE \Rightarrow \sqrt{\frac{\chi^2_{n-1,\alpha/2}}{n-1}}\left(\hat{C}pl,\ \hat{C}pu\right) + \frac{z_{\alpha/2}}{\sqrt{9n}}(1,\ -1). \tag{3.128}$$

We now see how to form a $(1 - \alpha)^2$ joint confidence region for the true (Cpl, Cpu). To get the two upper vertices, we start at the point estimator $\left(\hat{C}pl, \hat{C}pu\right)$. We "inflate" it by the factor

$\sqrt{\frac{\chi^2_{n-1,1-\alpha/2}}{n-1}}$, then we walk due northwest and due southeast $\frac{z_{\alpha/2}}{\sqrt{9n}}\sqrt{2}$ units to arrive at the points marked *SW* and *SE*. To get the two lower vertices, we start at $(\hat{C}pl, \hat{C}pu)$, "deflate" it by the factor $\sqrt{\frac{\chi^2_{n-1,\alpha/2}}{n-1}}$, then we walk due northwest and due southeast by the *same* $\frac{z_{\alpha/2}}{\sqrt{9n}}\sqrt{2}$ units to arrive at the points marked *NW* and *NE*. We therefore have a $(1 - \alpha)^2$ joint confidence region for the true (*Cpl*, *Cpu*) in the shape of a parallelogram. The parallelogram is a rectangle if and only if the point estimator $(\hat{C}pl, \hat{C}pu)$ lies on the ray of potentiality, that is, if and only if $(\hat{C}pl, \hat{C}pu) = (\hat{C}p, \hat{C}p)$. Only then will the $\frac{z_{\alpha/2}}{\sqrt{9n}}\sqrt{2}$ "walk" from the subspace spanned by $(\hat{C}pl, \hat{C}pu)$ be orthogonal to the subspace spanned by $(\hat{C}pl, \hat{C}pu)$. Furthermore, it is clear that the *SW* and *SE* vertices project into the same point on the ray of potentiality, while the *NW* and *NE* vertices project into a second, but lower, common point on the ray. These two projections are respectively, the upper and lower $(1 - \alpha)$ confidence limits for the true *Cp*. This is not surprising. The (*Cpl*, *Cpu*) joint confidence region was born of the (μ, σ) joint confidence region, the construction of which began with an unconditional confidence interval for σ. And *Cp* depends on σ but not μ.

We hasten to add that the $(1 - \alpha)^2$ confidence region thus constructed is only one of many possible $(1 - \alpha)^2$ confidence regions. There are uncountable possibilities. This region is perhaps the easiest to construct. We must also state that the parallelogram as drawn in Figure 3.4 has a general orientation between 45 and 90 degrees from the horizontal *Cpl* axis. This will be the case whenever $\hat{C}pl < \hat{C}pu$. On the other hand, the orientation will be between 0 and 45 degrees from horizontal whenever $\hat{C}pl > \hat{C}pu$. When $\hat{C}pl = \hat{C}pu = \hat{C}p$, the confidence region is a rectangle lying at an exact 45 degree orientation to horizontal, on the ray of potentiality.

To take a numerical example, suppose we have estimated

$$\left(\hat{C}pl,\quad \hat{C}pu\right) = (1.200,\quad 1.300) \tag{3.129}$$

based on $n = 101$. For $\alpha = 0.05$,

$$\sqrt{\frac{\chi^2_{n-1,\,1-\alpha/2}}{n-1}} = \sqrt{\frac{\chi^2_{100,\,0.975}}{100}} \approx \sqrt{\frac{129.561}{100}} \approx 1.138248655,$$

$$\sqrt{\frac{\chi^2_{n-1,\,\alpha/2}}{n-1}} = \sqrt{\frac{\chi^2_{100,\,0.025}}{100}} \approx \sqrt{\frac{74.2219}{100}} \approx 0.861521329,$$

$$\frac{z_{\alpha/2}}{\sqrt{9n}} = \frac{z_{0.025}}{\sqrt{9(101)}} \approx \frac{1.96}{\sqrt{909}} \approx 0.065009096.$$

Substituting into equations (3.125) through (3.128) gives

$$SW \Rightarrow 1.138248655(1.200,\,1.300) + 0.065009096(-1,\,1) \approx (1.301,\,1.545),$$

$$SE \Rightarrow 1.138248655(1.200,\,1.300) + 0.065009096(1,\,-1) \approx (1.431,\,1.415),$$

$$NW \Rightarrow 0.861521329(1.200,\,1.300) + 0.065009096(-1,\,1) \approx (0.969,\,1.185),$$

$$NE \Rightarrow 0.861521329(1.200,\,1.300) + 0.065009096(1,\,-1) \approx (1.099,\,1.055), \tag{3.130}$$

which are the four vertices of a 0.9025 joint confidence parallelogram for the true $(Cpl,\ Cpu)$, displayed as Figure 3.5. Note that

$$\frac{1.301 + 1.545}{2} = \frac{1.431 + 1.415}{2} = 1.423,$$

$$\frac{0.969 + 1.185}{2} = \frac{1.099 + 1.055}{2} = 1.077,$$

that is, the SW and SE vertices project into the upper limit of the previously computed 0.95 confidence interval for σ, while the NW and NE vertices project into the lower limit of the confidence interval.

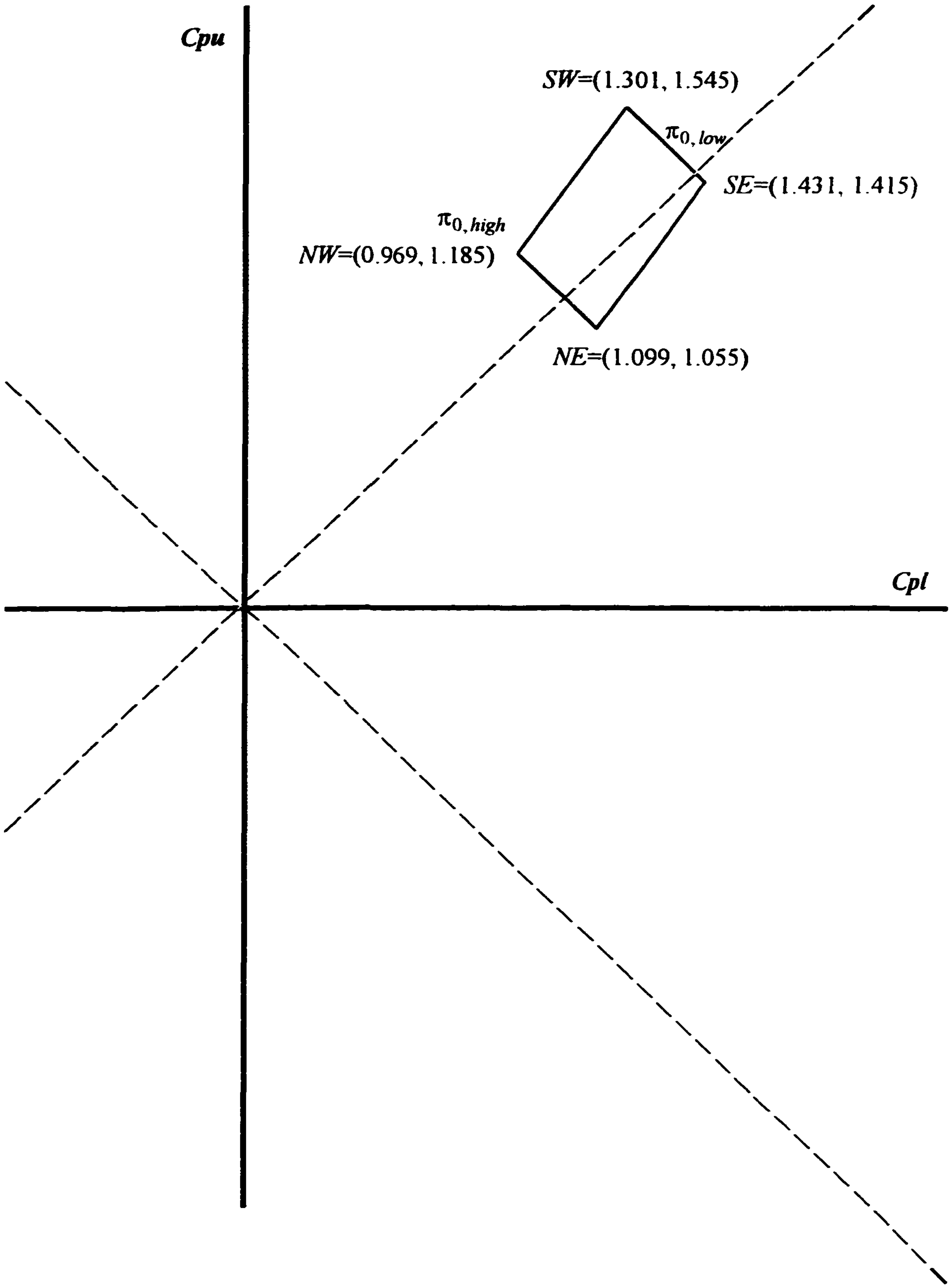

Figure 3.5. 0.9025 Joint Confidence Region for True (*Cpl*, *Cpu*), Unknown (μ, σ)

Determining a $(1-\alpha)^2$ confidence interval for the true proportion π_0 of product outside specification from the (*Cpl*, *Cpu*) confidence region is a simple task. We ask the reader to consider Figure 3.5 in order to address the sensitivity of π_0 to movement in the (*Cpl*, *Cpu*) plane.

Now every feasible point (*Cpl*, *Cpu*) of Figure 3.5 maps into a value π_0. We may imagine a bivariate function

$$\pi_0(Cpl, Cpu) = \Phi[-3Cpl] + \Phi[-3Cpu] = \pi_0$$

defined for feasible (*Cpl*, *Cpu*). Starting at any fixed feasible point (C^*pl, C^*pu), then moving orthogonally toward the ray of potentiality, decreases π_0. This corresponds to moving the process mean μ toward the specification interval midpoint *m* while holding the process standard deviation σ fixed. On the other hand, starting at any positive fixed feasible point (C^*pl, C^*pu), then moving away from the origin in the subspace spanned by (C^*pl, C^*pu), decreases π_0. This corresponds to decreasing the process standard deviation σ while holding the process mean μ fixed. There are problems when one of the pair (C^*pl, C^*pu) is negative. This corresponds to the mean μ being outside the specification interval. In such a case, one actually *increases* the proportion π_0 of product outside specification by decreasing σ. However, in our numerical example, all feasible (*Cpl*, *Cpu*) are positive and it is a simple matter to locate the two (*Cpl*, *Cpu*) points which determine the endpoints of a $(1-\alpha)^2$ confidence interval for the true π_0. Clearly, the intersection of the line segment joining the *SW* and *SE* vertices with the ray of potentiality determines the lower endpoint of this confidence interval, while the *NW* vertex determines the upper endpoint. These are the points in Figure 3.5,

$$\pi_{0,high} = (0.969,\quad 1.185) \quad \text{and} \quad \pi_{0,low} = (1.423,\quad 1.423). \tag{3.131}$$

Since

$$\Phi[-3(0.969)] + \Phi[-3(1.185)] = \Phi[-2.907] + \Phi[-3.555]$$

$$= 0.0018246 + 0.0001890 = 0.0020136 \tag{3.132}$$

and

$$\Phi[-3(1.423)] + \Phi[-3(1.423)] = \Phi[-4.269] + \Phi[-4.269]$$

$$= 0.0000098 + 0.0000098 = 0.0000196, \tag{3.133}$$

a 0.9025 confidence interval for the true proportion π_0 of product outside specification is given by

$$[0.0000196, \quad 0.0020136]. \tag{3.134}$$

We can get the associated 0.95 confidence interval for a *potential* π_0. Since

$$2\Phi[-3(1.077)] = 2\Phi[-3.231] = 2(0.0006168) = 0.0012336, \tag{3.135}$$

$$2\Phi[-3(1.423)] = 2\Phi[-4.269] = 2(0.0000098) = 0.0000196, \tag{3.136}$$

we have

$$[0.0000196, \quad 0.0012336]. \tag{3.137}$$

Observe that the lower limit of this 0.95 confidence interval for potential π_0 falls at the lower limit of the 0.9025 confidence interval for the current π_0, as it should. Also, the upper limit of this 0.95 confidence interval for potential π_0 falls below the upper limit of the 0.9025 confidence interval for the current π_0, as it should.

Note that the *NW* vertex, in isolation, locates a $(1 - \alpha/2)(1 - \alpha)$ upper confidence bound on the true current π_0. This is because the triangle with vertices *NW*, *NE*, and $\overline{X}$ of Figure 3.3 is a $(1 - \alpha/2)(1 - \alpha)$ confidence region for the true (μ, σ).

While the *Cpk* index is not of primary interest to us, we see how to arrive at a $(1 - \alpha)^2$ confidence interval for the true *Cpk* from Figure 3.5. First we recognize that for all $(1 - \alpha)^2$-plausible (*Cpl*, *Cpu*) *above* the ray of potentiality, we have *Cpk* = *Cpl*, putting the plausible *Cpk* in the interval [0.969, 1.423]. Also, for all $(1 - \alpha)^2$-plausible (*Cpl*, *Cpu*) *below* the ray of potentiality, *Cpk* = *Cpu*, putting the plausible *Cpk* in the interval [1.055, 1.423]. Taken together, we have a 0.9025 confidence interval for the true *Cpk* of [0.969, 1.423]. Recalling the 0.95 confidence interval for *Cp* to be [1.077, 1.423], and the identity

$$\pi_0 = \Phi[-3Cpl] + \Phi[-3Cpu] = \Phi[-3Cpk] + \Phi[-3(2Cp - Cpk)],$$

we have

$$\begin{aligned}\Phi[-3(0.969)] + \Phi[-3(2(1.077) - 0.969)] &= \Phi[-2.907] + \Phi[-3.555] \\ &= 0.0018246 + 0.0001890 = 0.0020136 \qquad (3.138)\end{aligned}$$

and

$$\begin{aligned}\Phi[-3(1.423)] + \Phi[-3(2(1.423) - 1.423)] &= \Phi[-4.269] + \Phi[-4.269] \\ &= 0.0000098 + 0.0000098 = 0.0000196, \qquad (3.139)\end{aligned}$$

giving a 0.9025 confidence interval for the true proportion π_0 of product outside specification,

$$[0.0000196, \quad 0.0020136]. \qquad (3.140)$$

This is identical to the confidence interval for π_0 previously derived from (*Cpl*, *Cpu*), given as equation (3.134). It is obvious that a confidence interval for π_0 is much more easily gotten from (*Cpl*, *Cpu*) than from (*Cp*, *Cpk*).

We must point out that our method of determining a $(1 - \alpha)^2$ joint confidence interval for the true (*Cpl*, *Cpu*) is considerably easier than the comparable methods of Levinson (1997a). He must utilize noncentral *t* tables. We use *z* and chi-squared tables.

3.7. Estimating *Cpk* when Both μ and σ are Unknown

Chou *et al.* (1989) derive the probability density function and moments of *Cpk*. However, we cannot resist an alternate derivation of the mean in a parameterization which provides more direct insight into this average behavior of *Cpk*.

The index *Cpk* is given by

$$Cpk = \min\{Cpl, Cpu\} = \min\left\{\frac{\mu - LSL}{3\sigma}, \frac{USL - \mu}{3\sigma}\right\}. \tag{3.141}$$

It provides a numerical indication of the capability of a process that is not centered between its specification limits. When the process is centered, the *Cpk* index is numerically identical to the *Cp* index. The natural estimator of *Cpk* is given by

$$\hat{C}pk = \min\{\hat{C}pl, \hat{C}pu\} = \min\left\{\frac{\overline{X} - LSL}{3S}, \frac{USL - \overline{X}}{3S}\right\}, \tag{3.142}$$

where

$$(\overline{X}, S) = \left(\frac{1}{n}\sum_{i=1}^{n} X_i,\ \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2}\right). \tag{3.143}$$

As before, it will be fruitful to investigate the random variable $\hat{C}pk/Cpk$. In what follows, we cannot allow a *Cpk* of zero. In other words, we will not allow μ to be at either *LSL* or *USL*, the specification limits. These poles must be analyzed separately. We have

$$\frac{\hat{C}pk}{Cpk} = \frac{\min\left\{\frac{\overline{X} - LSL}{3S}, \frac{USL - \overline{X}}{3S}\right\}}{\min\left\{\frac{\mu - LSL}{3\sigma}, \frac{USL - \mu}{3\sigma}\right\}}. \tag{3.144}$$

It is seen that this random variable depends, of course, on μ in the parameter space, but also on the realization of the sample mean $\overline{X}$ above or below the midpoint *m* of the specification interval [*LSL*, *USL*].

Consider the region of the parameter space where $\mu \le m$. If also $\bar{X} \le m$, then

$$\frac{\hat{C}pk}{Cpk} = \frac{\bar{X} - LSL}{3S} \frac{3\sigma}{\mu - LSL} = \frac{\bar{X} - LSL}{\mu - LSL} \frac{\sigma}{S}, \tag{3.145}$$

but if $\bar{X} > m$, then

$$\frac{\hat{C}pk}{Cpk} = \frac{USL - \bar{X}}{3S} \frac{3\sigma}{\mu - LSL} = \frac{USL - \bar{X}}{\mu - LSL} \frac{\sigma}{S}. \tag{3.146}$$

We can get the expected value of the ratio $\hat{C}pk/Cpk$ in the region of the parameter space where $\mu \le m$ by using the law of total probability applied to expectations, that is,

$$\begin{aligned} E\left[\frac{\hat{C}pk}{Cpk}\right] &= E\left[\frac{\bar{X} - LSL}{\mu - LSL}\frac{\sigma}{S}\middle| \bar{X} \le m\right]\Pr\left[\bar{X} \le m\right] + E\left[\frac{USL - \bar{X}}{\mu - LSL}\frac{\sigma}{S}\middle| \bar{X} > m\right]\Pr\left[\bar{X} > m\right] \\ &= E\left[\frac{\sigma}{S}\right]\left\{E\left[\frac{\bar{X} - LSL}{\mu - LSL}\middle| \bar{X} \le m\right]\Pr\left[\bar{X} \le m\right] + E\left[\frac{USL - \bar{X}}{\mu - LSL}\middle| \bar{X} > m\right]\Pr\left[\bar{X} > m\right]\right\}, \end{aligned} \tag{3.147}$$

the second line following from the independence of $(\bar{X}, S)$. Recall that

$$E\left[\frac{\sigma}{S}\right] = \frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}}. \tag{3.148}$$

Now it is known that

$$E\left[\bar{X} \middle| \bar{X} \le m\right] = \mu - \frac{\sigma}{\sqrt{n}}\left\{\frac{\phi}{\Phi}\right\} \tag{3.149}$$

and

$$E\left[\bar{X} \middle| \bar{X} > m\right] = \mu + \frac{\sigma}{\sqrt{n}}\left\{\frac{\phi}{1-\Phi}\right\}, \tag{3.150}$$

where ϕ and Φ are the standard normal pdf and cdf evaluated at $\dfrac{m-\mu}{\sigma/\sqrt{n}}$. Substituting into equation (3.147) gives

$$E\left[\frac{\hat{C}pk}{Cpk}\right] = \frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}}$$

$$\times\left\{\frac{-LSL+\mu-\frac{\sigma}{\sqrt{n}}\left\{\frac{\phi}{\Phi}\right\}}{\mu-LSL}\Phi+\frac{USL-\left\{\mu+\frac{\sigma}{\sqrt{n}}\left\{\frac{\phi}{1-\Phi}\right\}\right\}}{\mu-LSL}(1-\Phi)\right\}$$

$$=\frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}}\left\{\Phi-\frac{\phi\sigma/\sqrt{n}}{\mu-LSL}+\frac{USL-\mu}{\mu-LSL}(1-\Phi)-\frac{\phi\sigma/\sqrt{n}}{\mu-LSL}\right\}$$

$$=\frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}}\left\{\Phi+\frac{USL-\mu}{\mu-LSL}(1-\Phi)-2\frac{\phi\sigma/\sqrt{n}}{\mu-LSL}\right\}. \qquad (3.151)$$

Now consider the region of the parameter space where $\mu > m$. If also $\overline{X} \le m$, then

$$\frac{\hat{C}pk}{Cpk}=\frac{\overline{X}-LSL}{3S}\frac{3\sigma}{USL-\mu}=\frac{\overline{X}-LSL}{USL-\mu}\frac{\sigma}{S}, \qquad (3.152)$$

but if $\overline{X} > m$, then

$$\frac{\hat{C}pk}{Cpk}=\frac{USL-\overline{X}}{3S}\frac{3\sigma}{USL-\mu}=\frac{USL-\overline{X}}{USL-\mu}\frac{\sigma}{S}. \qquad (3.153)$$

We can get the expected value of the ratio $\hat{C}pk/Cpk$ in the region of the parameter space where $\mu > m$ by using the law of total probability applied to expectations, that is,

$$E\left[\frac{\hat{C}pk}{Cpk}\right]=E\left[\frac{\overline{X}-LSL}{USL-\mu}\frac{\sigma}{S}\Big|\overline{X}\le m\right]\Pr\left[\overline{X}\le m\right]+E\left[\frac{USL-\overline{X}}{USL-\mu}\frac{\sigma}{S}\Big|\overline{X}>m\right]\Pr\left[\overline{X}>m\right]$$

$$=E\left[\frac{\sigma}{S}\right]\left\{E\left[\frac{\overline{X}-LSL}{USL-\mu}\Big|\overline{X}\le m\right]\Pr\left[\overline{X}\le m\right]+E\left[\frac{USL-\overline{X}}{USL-\mu}\Big|\overline{X}>m\right]\Pr\left[\overline{X}>m\right]\right\},$$

(3.154)

the second line following from the independence of $(\overline{X}, S)$. Substituting into equation (3.154) gives

$$E\left[\frac{\hat{C}pk}{Cpk}\right]=\frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}}$$

$$\times\left\{\frac{-LSL+\mu-\frac{\sigma}{\sqrt{n}}\left\{\frac{\phi}{\Phi}\right\}}{USL-\mu}\Phi+\frac{USL-\left\{\mu+\frac{\sigma}{\sqrt{n}}\left\{\frac{\phi}{1-\Phi}\right\}\right\}}{USL-\mu}(1-\Phi)\right\}$$

$$=\frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}}\left\{\frac{\mu-LSL}{USL-\mu}\Phi-\frac{\phi\sigma/\sqrt{n}}{USL-\mu}+(1-\Phi)-\frac{\phi\sigma/\sqrt{n}}{USL-\mu}\right\}$$

$$=\frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}}\left\{\frac{\mu-LSL}{USL-\mu}\Phi+(1-\Phi)-2\frac{\phi\sigma/\sqrt{n}}{USL-\mu}\right\}, \tag{3.155}$$

where ϕ and Φ are the standard normal pdf and cdf evaluated at $\frac{m-\mu}{\sigma/\sqrt{n}}$.

We now have two expressions for $E\left[\hat{C}pk/Cpk\right]$,

$$E\left[\frac{\hat{C}pk}{Cpk}\right]=\frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}}\left\{\Phi+\frac{USL-\mu}{\mu-LSL}(1-\Phi)-2\frac{\phi\sigma/\sqrt{n}}{\mu-LSL}\right\} \quad \text{for} \quad \mu\le m$$

and

$$E\left[\frac{\hat{C}pk}{Cpk}\right]=\frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}}\left\{\frac{\mu-LSL}{USL-\mu}\Phi+(1-\Phi)-2\frac{\phi\sigma/\sqrt{n}}{USL-\mu}\right\} \quad \text{for} \quad \mu>m,$$

$$\tag{3.156}$$

where ϕ and Φ are the standard normal pdf and cdf evaluated at $\frac{m-\mu}{\sigma/\sqrt{n}}$.

We can reparameterize these two expressions into a single expression for $E\left[\hat{C}pk/Cpk\right]$. Let $\Delta=\frac{USL-LSL}{\sigma}$ be the length of the specification interval in process standard deviations and let $\delta=\left|\frac{m-\mu}{\sigma}\right|$ be the unsigned "offset" distance between the process mean and the specification interval midpoint in process standard deviations. We then have the single expression for $E\left[\hat{C}pk/Cpk\right]$,

$$E\left[\frac{\hat{C}pk}{Cpk}\right]=\frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}}\left\{1+4\left\{\frac{\delta(1-\Phi)-\phi/\sqrt{n}}{\Delta-2\delta}\right\}\right\}, \tag{3.157}$$

where ϕ and Φ are the standard normal pdf and cdf evaluated at $\left|\frac{m-\mu}{\sigma/\sqrt{n}}\right| = \left|\frac{m-\mu}{\sigma}\right|\sqrt{n} = \delta\sqrt{n}$.

We derive this single expression in Appendix A. We point out that this equation (3.157) is exact and not an approximation.

Consider δ fixed and not equal to zero. We see that $\phi\left[\delta\sqrt{n}\right]$ approaches zero, while $\Phi\left[\delta\sqrt{n}\right]$ approaches one, as n approaches infinity. We have

$$\lim_{n\to\infty} \frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}}\left\{1+4\left\{\frac{\delta(1-\Phi)-\phi/\sqrt{n}}{\Delta-2\delta}\right\}\right\}$$

$$= 1\left\{1+4\left\{\frac{\delta(1-1)-0}{\Delta-2\delta}\right\}\right\} = 1. \tag{3.158}$$

That is, $E\left[\hat{C}pk/Cpk\right]$ approaches one as n grows large. Therefore, $\hat{C}pk$ is asymptotically unbiased for Cpk.

On the other hand, considering δ fixed and equal to zero, we have for each finite n,

$$E\left[\frac{\hat{C}pk}{Cpk}\right] = \frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}}\left\{1-\frac{4}{\Delta\sqrt{2\pi n}}\right\}, \tag{3.159}$$

which approaches one as n grows large, and again $\hat{C}pk$ is asymptotically unbiased for Cpk.

Equation (3.157) reveals the high degree of complexity in the structure of $E\left[\hat{C}pk/Cpk\right]$, attributable to the expression $\frac{\delta(1-\Phi)-\phi/\sqrt{n}}{\Delta-2\delta}$, which is not monotone in n. Kotz and Johnson (1993) give the values of $E\left[\hat{C}pk/Cpk\right]$ for $\mu = m$ and $\Delta = 6$, which corresponds to Cp and Cpk each equal to one. They compute a $E\left[\hat{C}pk/Cpk\right] = E\left[\hat{C}pk\right]$ equal to 1.002 for n = 10, decreasing to 0.977 for $n = 30$, and increasing to 1.000 for $n = 79500$. It would appear that the complex behavior of $\hat{C}pk$ is not a strong argument for its use as a measure of process capability.

CHAPTER 4. CAPABILITY INDICES UNDER NORMAL CORRELATION

The previous models have, as a key assumption, independent, identically distributed normal random variates. The situation changes substantially when the variates are no longer independent. In particular, serial autocorrelation is not an uncommon feature of a sample taken over time. It is known that the sample variance is a biased estimator of the population variance when the sample observations are correlated. The bias can be positive or negative and substantial in size. Yang and Hancock (1990) show that for an autocorrelated sample $\{X_i\}_n$ identically distributed from a population with mean μ and variance σ^2, the usual sample variance $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ has expected value $ES^2 = (1-\bar{\rho})\sigma^2$, where

$$\bar{\rho} = \frac{1}{n(n-1)}\sum_{i \neq j}^{n(n-1)} \rho_{ij} = \frac{2}{n(n-1)}\sum_{i<j}^{n(n-1)/2} \rho_{ij} \tag{4.1}$$

is the average of the $n(n-1)$ pairwise correlation parameters of the model. This implies that in $\hat{C}p = \frac{USL - LSL}{6S}$, the denominator will tend to underestimate σ and so $\hat{C}p$ will tend to overestimate Cp. This reinforces the effect of the bias in $1/S$ as an estimator of $1/\sigma$ which also tends to produce overestimation of Cp. There will be a similar effect for Cpk.

We confront two unpleasant facts throughout this chapter. First, with the sample $\{X_i\}_n$ autocorrelated, the random variable σ/S is no longer distributed as $\sqrt{n-1}\,\chi_{n-1}^{-1}$ since $\{X_i\}_n$ are no longer independent. Second, $\bar{X}$ and $\sum_{i=1}^{n}(X_i - \bar{X})^2$ are dependent. To see this, let $\mathbf{x}$ be a random n-vector with mean vector μ with identical coordinates, correlation matrix $\mathbf{R}$, and covariance matrix $\sigma^2\mathbf{R}$. Let $\mathbf{C} = \mathbf{I} - \frac{1}{n}\mathbf{1}\mathbf{1}'$ be the centering matrix, that is,

$$\mathbf{Cx} = \mathbf{x} - \bar{\mathbf{x}} = (X_1 - \bar{X}, X_2 - \bar{X}, \cdots, X_n - \bar{X})' \tag{4.2}$$

and

$$\mathbf{x}'\mathbf{C}\mathbf{x} = \mathbf{x}'\mathbf{C}\mathbf{C}\mathbf{x} = (\mathbf{x} - \bar{\mathbf{x}})'(\mathbf{x} - \bar{\mathbf{x}}) = \sum_{i=1}^{n} \left(X_i - \bar{X} \right)^2 . \tag{4.3}$$

Then

$$\mathbf{C}\mathbf{x} = \mathbf{x} - \bar{\mathbf{x}} \sim N_n\left(\mathbf{C}\boldsymbol{\mu}, \sigma^2 \mathbf{CRC} \right) = N_n\left(\mathbf{0}, \sigma^2 \mathbf{CRC} \right) \tag{4.4}$$

and

$$(\mathbf{I} - \mathbf{C})\mathbf{x} = \bar{\mathbf{x}} \sim N_n\left((\mathbf{I} - \mathbf{C})\boldsymbol{\mu}, \sigma^2 (\mathbf{I} - \mathbf{C})\mathbf{R}(\mathbf{I} - \mathbf{C}) \right) = N_n\left(\boldsymbol{\mu}, \sigma^2 (\mathbf{I} - \mathbf{C})\mathbf{R}(\mathbf{I} - \mathbf{C}) \right). \tag{4.5}$$

We point out that $\mathbf{C}\mathbf{x} = \mathbf{x} - \bar{\mathbf{x}}$ is degenerate in the $(n - 1)$ dimensional hyperplane defined by the condition $(\mathbf{x} - \bar{\mathbf{x}})'\mathbf{1} = 0$, while $(\mathbf{I} - \mathbf{C})\mathbf{x} = \bar{\mathbf{x}}$ is degenerate in the line spanned by the vector of ones, $\mathbf{1}$.

Now since

$$(\mathbf{I} - \mathbf{C})\sigma^2 \mathbf{RC} = \sigma^2 \mathbf{RC} - \sigma^2 \mathbf{CRC} \neq \mathbf{0}_{n \times n}, \tag{4.6}$$

we have by a theorem in Graybill (1976), page 138, that the linear form $\bar{\mathbf{x}} = (\mathbf{I} - \mathbf{C})\mathbf{x}$ and the quadratic form $(\mathbf{x} - \bar{\mathbf{x}})'(\mathbf{x} - \bar{\mathbf{x}}) = \mathbf{x}'\mathbf{C}\mathbf{x}$ are dependent. That is, $\bar{X}$ and $\sum_{i=1}^{n} (X_i - \bar{X})^2$ are dependent. This prevents us from factoring expectations such as

$$E\left[\frac{f(\bar{X})}{g(S)} \right] = E\left[f(\bar{X}) \right] E\left[\frac{1}{g(S)} \right], \tag{4.7}$$

which would simplify certain derivations for well-behaved f and g.

4.1. Why Some Real World Data Exhibit Autocorrelation

In this section, we seek an explanation for why autocorrelated data are so often observed. What is the nature of the true random process which begets such realizations? We find a partial answer in the theory of linear stochastic differential equations. We must add that the major portion of this section is drawn from the textbook of Pandit and Wu (1983).

Many important problems in engineering, the physical sciences, and the social sciences, are initially formulated as differential equation models. The particular differential equations are usually suggested by a combination of existing theory and past experimental data. Perhaps the most familiar example to students of physics is Newton's law.

$$m\frac{d^2X(t)}{dt^2} = F\left[t, X(t), \frac{dX(t)}{dt}\right]. \tag{4.8}$$

for the position $X(t)$ of a particle acted on by a force F which may be a function of time t, the position $X(t)$, and the velocity $dX(t)/dt$. To determine the motion of a particle acted on by a given force F, it is necessary to find a function $X(t)$ satisfying equation (4.8). If the force is that due to gravity, then

$$m\frac{d^2X(t)}{dt^2} = -mg. \tag{4.9}$$

On integrating equation (4.9), we have

$$\frac{dX(t)}{dt} = -gt + c_1$$

$$X(t) = -\tfrac{1}{2}gt^2 + c_1t + c_2, \tag{4.10}$$

where c_1 and c_2 are constants. To determine $X(t)$ completely, it is necessary to specify two additional conditions, such as the position and velocity of the particle at some instant of time. These conditions can be used to determine the constants c_1 and c_2.

An even simpler model is the constant percentage decay model,

$$\frac{dX(t)}{dt} = -\alpha_0X(t)$$

or

$$\frac{dX(t)}{dt}+\alpha_0 X(t)=0. \tag{4.11}$$

Its solution is

$$X(t)=C_0\exp\{-\alpha_0 t\} \tag{4.12}$$

with $X(0)=C_0$ as boundary condition. This model postulates that at any time t, the characteristic X is instantaneously diminishing at a constant percentage rate α_0. This model may adequately describe the size of a physical object which exhibits a constant percentage decay at rate α_0.

Equations (4.9) and (4.11) describe models which are deterministic, that is, their future is assumed completely determined or predictable, with no error, from the past. More realistic models are so-called stochastic or probabilitistic models, which seek only to describe the long-run average behavior of the state variable X. We can convert equation (4.11) to a stochastic equation by adding the white noise forcing function $Z(t)$, giving

$$\frac{dX(t)}{dt}+\alpha_0 X(t)=Z(t)$$

or

$$(D+\alpha_0)X(t)=Z(t), \tag{4.13}$$

where $E[Z(t)]=0$ for all t and $E[Z(t)Z(t-u)]=\sigma_z^2\delta(u)$ for all t and u. This linear stochastic differential equation, called a continuous autoregressive model of order one, is denoted A(1).

To obtain the solution to equation (4.13), we invert the operator to get

$$X(t)=(D+\alpha_0)^{-1}Z(t).$$

Since the inverse of differentiation is integration, we get

$$X(t) = \int_0^{\infty} G(u) Z(t-u)\, du = \int_0^{\infty} G(t-u) Z(u)\, du. \tag{4.14}$$

The solution to $G(t)$ in (4.14) is

$$G(t) = \exp\{-\alpha_0 t\} \tag{4.15}$$

for $t \geq 0$ and $G(t) = 0$ otherwise.

Therefore,

$$X(t) = \int_0^{\infty} \exp\{-\alpha_0 u\} Z(t-u)\, du = \int_0^{\infty} \exp\{-\alpha_0 (t-u)\} Z(u)\, du. \tag{4.16}$$

Consider now $\gamma(s) = E[X(t)X(t-s)]$, the process autocovariance function. We have

$$\gamma(s) = \int_0^{\infty} G(u) G(u+s)\, du.$$

So we have for the A(1) model,

$$\gamma(s) = \int_0^{\infty} \exp\{-\alpha_0 u\} \exp\{-\alpha_0 (u+s)\}\, du$$

$$= \frac{\sigma_Z^2}{2\alpha_0} \exp\{-\alpha_0 s\} \tag{4.17}$$

for all nonnegative s. Note that this autocovariance function is never zero. Even though the forcing function $Z(t)$ is completely uncorrelated, the characteristic $X(t)$ exhibits nonzero autocovariance and nonzero autocorrelation at all lags s.

Many systems met in practice are continuous. For such systems, continuous models in the form of differential equations provide a "live" description. These continuous models are well-suited for characterization and they are therefore extremely useful in system analysis and system design, in addition to system prediction and control. On the other hand, discrete models are popular since they can be readily obtained from a discrete set of data by extending the linear regression methods. Moreover, such models are adequate when one is interested only in predicting a system at discrete points. With this in mind, we look for a discrete model which matches the continuous model on their autocovariance functions.

Let Δ be the fixed sampling interval. We have

$$\gamma(s) = E[X(t)X(t-s)],$$

and so

$$\gamma(k\Delta) = \gamma_k = E[X(t)X(t-k\Delta)] = E[X_t X_{t-k}].$$

Now since

$$\gamma(s) = \frac{\sigma_Z^2}{2\alpha_0}\exp\{-\alpha_0 s\}$$

and $\gamma_k = \gamma(k\Delta)$, we have

$$\gamma(s) = \gamma(k\Delta) = \frac{\sigma_Z^2}{2\alpha_0}\exp\{-\alpha_0 s\} = \frac{\sigma_Z^2}{2\alpha_0}\phi^k,$$

where

$$\phi = \exp\{-\alpha_0\Delta\}. \tag{4.18}$$

To derive an expression for the variance of the error term, we note from the Yule-Walker equations that

$$\begin{cases}\gamma_0 = \phi\gamma_1 + \sigma_a^2 \\ \gamma_k = \phi\gamma_{k-1} & , \ k \geq 1\end{cases}$$

implying that

$$\sigma_a^2 = \frac{\sigma_Z^2}{2\alpha_0}\left(1-\phi^2\right) = \frac{\sigma_Z^2}{2\alpha_0}\left(1-\exp\{-2\alpha_0\Delta\}\right). \tag{4.19}$$

Equations (4.18) and (4.19) give the discrete AR(1) parameters $\left(\phi,\ \sigma_a^2\right)$ in terms of the continuous A(1) parameters $\left(\alpha_0,\ \sigma_z^2\right)$ and the sampling interval Δ. We can invert these equations to get $\alpha_0 = \frac{-\ln\phi}{\Delta}$ and $\sigma_z^2 = \frac{2\alpha_0\sigma_a^2}{1-\phi^2} = \frac{-2\ln\phi}{\Delta}\frac{\sigma_a^2}{1-\phi^2}$. We display these two important sets of relations together as

$$\phi = \exp\{-\alpha_0\Delta\},\quad \sigma_a^2 = \frac{\sigma_Z^2}{2\alpha_0}\left(1-\phi^2\right) = \frac{\sigma_Z^2}{2\alpha_0}(1-\exp\{-2\alpha_0\Delta\}) \tag{4.20}$$

and

$$\alpha_0 = \frac{-\ln\phi}{\Delta},\quad \sigma_z^2 = \frac{2\alpha_0\sigma_a^2}{1-\phi^2} = \frac{-2\ln\phi}{\Delta}\frac{\sigma_a^2}{1-\phi^2}. \tag{4.21}$$

We can study the effect of the length Δ of the sampling interval on the sampled discrete model by observing equation (4.18). First consider the case of a large sampling interval Δ. When the sampling interval is large, we see from equation (4.18) that

$$\lim_{\Delta\to\infty}\phi = \lim_{\Delta\to\infty}\exp\{-\alpha_0\Delta\} = 0, \tag{4.22}$$

and so the model becomes $X_t = a_t$, an AR(0) model or a discrete white noise sequence with variance, again by equation (4.19),

$$\lim_{\Delta\to\infty}\sigma_a^2 = \lim_{\Delta\to\infty}\frac{\sigma_Z^2}{2\alpha_0}\left(1-\phi^2\right) = \frac{\sigma_Z^2}{2\alpha_0} = \gamma_0. \tag{4.23}$$

This shows that if the observations are taken so far apart that the autocovariance or autocorrelation function decays to zero within the length of the sampling interval, then there is practically no correlation between the successively sampled observations. Note that

$$\sigma_X^2 = \frac{\sigma_a^2}{1-\phi^2} = \frac{\sigma_Z^2}{2\alpha_0}\frac{1-\phi^2}{1-\phi^2} = \frac{\sigma_Z^2}{2\alpha_0} = \gamma(0). \tag{4.24}$$

When the sampling interval is small, we have from equation (4.18) that

$$\lim_{\Delta\to 0}\phi = \lim_{\Delta\to 0}\exp\{-\alpha_0\Delta\} = 1\ , \tag{4.25}$$

and so the model becomes $X_t - X_{t-1} = a_t$. This is the nonstationary random walk in discrete time.

The linear stochastic difference equation given by

$$\begin{aligned} X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} - \cdots - \phi_n X_{t-n} \\ = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_{n-1} a_{t-n+1} \end{aligned} \tag{4.26}$$

is a discrete autoregressive moving average model, denoted ARMA(n, n - 1). In an analogous manner, the linear stochastic differential equation given by

$$\begin{aligned} \frac{d^n X(t)}{dt} + \alpha_{n-1} \frac{d^{n-1} X(t)}{dt^{n-1}} + \cdots + \alpha_0 X(t) \\ = \beta_{n-1} \frac{d^{n-1} Z(t)}{dt^{n-1}} + \cdots + \beta_1 \frac{dZ(t)}{dt} + Z(t) \end{aligned} \tag{4.27}$$

is a continuous autoregressive moving average model, denoted AM(n, n - 1). The solution to the discrete model can be expressed as $X_t = \sum_{j=0}^{\infty} G_j a_{t-j}$, while the solution to the continuous model is $X(t) = \int_0^\infty G(v) Z(t-v) dv$. The function G is called Green's function. These are the ψ-weights in the discrete ARMA models of Box and Jenkins (1993). From Pandit and Wu (1983), we have the remarkable fact that *when a stochastic system governed by equation* (4.27) *is sampled at uniform intervals, the resultant discrete system has a representation of the same form as equation* (4.26).

The simplest system governed by differential equations with constant coefficients is the first order system, $\frac{dX(t)}{dt} + \alpha_0 X(t) = Z(t)$. Its uniformly sampled process has the discrete representation of the AR(1) model, $X_t - \phi_1 X_{t-1} = a_t$. It can be shown that this is no longer true for higher order systems. For example, a uniformly sampled second order system does not, in general, have an AR(2) representation. However, it can be shown that when a continuous autoregressive moving average process, in the form of a linear differential equation of autoregressive order n and *arbitrary* moving average order, is sampled at uniform intervals, the resultant sampled process is ARMA(n, n - 1). See Pandit and Wu (1983) for examples of some corresponding models.

4.2. The Indices (*Cpl*, *Cp*, *Cpu*) and the ARMA(p, q) Model

We have earlier demonstrated the equivalence of estimating the index (*Cpl*, *Cp*, *Cpu*) to estimating (μ, σ) whenever the data are *iid* normal realizations and the specification limits are known. We must now ask to what extent autocorrelated observations damage this equivalence.

Suppose that the random data obey a stationary normal ARMA(p, q) process, that is,

$$\left(1-\phi_1 B-\phi_2 B^2-\cdots-\phi_p B^p\right)\left(X_t-\mu\right)=\left(1-\theta_1 B-\theta_2 B^2-\cdots-\theta_q B^q\right)a_t$$

or

$$\phi_p(B)(X_t-\mu)=\theta_q(B)a_t, \tag{4.28}$$

where the a_t are *iid* $N\left(0,\sigma_a^2\right)$ and B is the backshift operator defined by $B^p X_t = X_{t-p}$. We can express $(X_t-\mu)$ in moving average form as

$$\begin{aligned}(X_t-\mu)&=\frac{\theta_q(B)}{\phi_p(B)}a_t=\psi(B)a_t\\&=\left(1+\psi_1 B+\psi_2 B^2+\psi_3 B^3+\cdots\right)a_t,\\&=a_t+\psi_1 a_{t-1}+\psi_2 a_{t-2}+\psi_3 a_{t-3}+\cdots\\&=\sum_{i=0}^{\infty}\psi_i a_{t-i},\end{aligned} \tag{4.29}$$

where $\psi_0 = 1$ and the sum converges in probability by assumption. The variance of X_t is given by

$$Var[X_t]=Var[X_t-\mu]=Var\left[\sum_{i=0}^{\infty}\psi_i a_{t-i}\right]$$

$$= \sum_{i=0}^{\infty} Var[\psi_i a_{t-i}] = \sum_{i=0}^{\infty} \psi_i^2 Var[a_{t-i}]$$

$$= \sigma_a^2 \sum_{i=0}^{\infty} \psi_i^2 . \tag{4.30}$$

Now marginally, each X_t is normal with mean μ and variance $\sigma_X^2 = \sigma_a^2 \sum_{i=0}^{\infty} \psi_i^2$. Therefore, the estimation of (Cpl, Cp, Cpu) must be equivalent to estimating $(\mu, \sigma_X) = \left(\mu, \quad \sigma_a \sqrt{\sum_{i=0}^{\infty} \psi_i^2}\right)$ for fixed LSL and USL.

Let us be clear here. If one is interested in characterizing a stable normal process for capability purposes, and if one defines a stable normal process as a series $\{X_t\}_{-\infty}^{+\infty}$ such that each X_t has identical marginal normal probability density function independent of time t, then admitting the stationary normal ARMA(p, q) processes impairs the indices (Cpl, Cp, Cpu) only to the extent that the variance parameter of X_t is now a function of several parameters. This makes its estimation a more difficult task. Of course, estimating the natural parameters (μ, σ_X) involves the same level of difficulty.

4.3. The Effect of Autocorrelation on the Sample Variance

Recall that for an autocorrelated sample $\{X_i\}_n$ identically distributed from a population with mean μ and variance σ^2, the usual sample variance $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ has expected value $ES^2 = (1-\bar{\rho})\sigma^2$, where

$$\bar{\rho} = \frac{1}{n(n-1)} \sum_{i \neq j}^{n(n-1)} \rho_{ij} = \frac{2}{n(n-1)} \sum_{i<j}^{n(n-1)/2} \rho_{ij} \tag{4.31}$$

is the average of the $n(n - 1)$ pairwise correlation parameters of the model. To see this,

$$\sum_{i=1}^{n}(X_i-\overline{X})^2=\sum_{i=1}^{n}X_i^2-\frac{1}{n}\left\{\sum_{i=1}^{n}X_i\right\}^2$$

$$=\sum_{i=1}^{n}X_i^2-\frac{1}{n}\left\{\sum_{i=1}^{n}X_i^2+\sum_{i\neq j}^{n(n-1)}X_iX_j\right\}$$

$$=\left\{1-\frac{1}{n}\right\}\sum_{i=1}^{n}X_i^2-\frac{1}{n}\sum_{i\neq j}^{n(n-1)}X_iX_j. \tag{4.32}$$

Taking expectations yields

$$E\left[\left\{1-\frac{1}{n}\right\}\sum_{i=1}^{n}X_i^2-\frac{1}{n}\sum_{i\neq j}^{n(n-1)}X_iX_j\right]$$

$$=\left\{1-\frac{1}{n}\right\}\sum_{i=1}^{n}E\left[X_i^2\right]-\frac{1}{n}\sum_{i\neq j}^{n(n-1)}E\left[X_iX_j\right]$$

$$=\left\{1-\frac{1}{n}\right\}n\left(\sigma^2+\mu^2\right)-\frac{1}{n}\left\{\sum_{i\neq j}^{n(n-1)}\sigma_{ij}+n(n-1)\mu^2\right\}$$

$$=\left\{1-\frac{1}{n}\right\}n\left(\sigma^2+\mu^2\right)-\frac{1}{n}\left\{n(n-1)\sigma^2\overline{\rho}+n(n-1)\mu^2\right\}$$

$$=(n-1)\left(\sigma^2+\mu^2\right)-(n-1)\sigma^2\overline{\rho}-(n-1)\mu^2$$

$$=(n-1)\sigma^2+(n-1)\mu^2-(n-1)\sigma^2\overline{\rho}-(n-1)\mu^2$$

$$=(n-1)\left(\sigma^2-\sigma^2\overline{\rho}\right)$$

$$=(n-1)(1-\overline{\rho})\sigma^2. \tag{4.33}$$

We see that

$$E\left[S^2\right]=E\left[\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\overline{X})^2\right]=\frac{1}{n-1}E\left[\sum_{i=1}^{n}(X_i-\overline{X})^2\right]$$

$$=\frac{1}{n-1}(n-1)(1-\overline{\rho})\sigma^2$$

$$=(1-\overline{\rho})\sigma^2, \tag{4.34}$$

and so we have the desired result, $ES^2=(1-\overline{\rho})\sigma^2$.

With a result from Searle (1982), it is even easier to demonstrate. Let $\mathbf{x}$ be a random n-vector with mean vector $\boldsymbol{\mu}$ with identical coordinates, correlation matrix $\mathbf{R}$, and covariance matrix $\sigma^2\mathbf{R}$. Let $\mathbf{C} = \mathbf{I} - \frac{1}{n}\mathbf{1}\mathbf{1}'$ be the centering matrix, that is,

$$\mathbf{Cx} = \mathbf{x} - \bar{\mathbf{x}} = \left(X_1 - \bar{X}, X_2 - \bar{X}, \cdots, X_n - \bar{X}\right)' \tag{4.35}$$

and

$$\mathbf{x}'\mathbf{Cx} = \mathbf{x}'\mathbf{CCx} = (\mathbf{x} - \bar{\mathbf{x}})'(\mathbf{x} - \bar{\mathbf{x}}) = \sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2. \tag{4.36}$$

From Searle (1982), we have

$$\begin{aligned} E[\mathbf{x}'\mathbf{Cx}] &= trace\left[\mathbf{C}\sigma^2\mathbf{R}\right] + \boldsymbol{\mu}'\mathbf{C}\boldsymbol{\mu} \\ &= \sigma^2 trace\mathbf{CR} + 0 \\ &= \sigma^2 trace\mathbf{CR} \\ &= (n-1)(1-\bar{\rho})\sigma^2. \end{aligned} \tag{4.37}$$

It follows that

$$E\left[S^2\right] = E\left[\frac{1}{n-1}\mathbf{x}'\mathbf{Cx}\right] = \frac{1}{n-1}E[\mathbf{x}'\mathbf{Cx}] = \frac{1}{n-1}(n-1)(1-\bar{\rho})\sigma^2 = (1-\bar{\rho})\sigma^2. \tag{4.38}$$

4.4. A Lower Bound

We can go further by observing that if one where to use the sample variance $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2$ as an estimator of the population variance σ^2, the ratio of the estimated Cp to the true process Cp would be

$$\frac{\hat{C}p}{Cp} = \frac{(USL - LSL)/6S}{(USL - LSL)/6\sigma} = \frac{\sigma}{S}. \tag{4.39}$$

Now since $(1-\bar{\rho}) = ES^2/\sigma^2$, we have

$$P = (1-\bar{\rho})^{-1/2} = \left\{ \frac{\sigma^2}{ES^2} \right\}^{1/2} = \frac{\sigma}{\sqrt{ES^2}}. \tag{4.40}$$

The ratio P is suspiciously close to $E[\sigma/S]$, but of course, they are not the same. However, by Jensen's Inequality, one can show that $P = \frac{\sigma}{\sqrt{ES^2}} < E\left[\frac{\sigma}{S}\right] = E\left[\frac{\hat{C}p}{Cp}\right]$, so we have a lower bound on the mean of the random variable $\hat{C}p/Cp$.

Jensen's Inequality states that if g is a convex function on the real line R, and both X and $g(X)$ are integrable random variables, then $g(E[X]) \le E[g(X)]$. Also, if g is a concave function on R, and both X and $g(X)$ are integrable random variables, then $g(E[X]) \ge E[g(X)]$. Strict convexity or concavity in the assumption implies strict inequality in the conclusion.

To get a lower bound on $E[\sigma/S]$, note that by strict concavity we have $E\sqrt{S^2} = E[S] < \sqrt{ES^2}$ or $1/E[S] > 1/\sqrt{ES^2}$. By strict convexity we have $E[1/S] > 1/E[S]$. Putting together gives $1/\sqrt{ES^2} < 1/E[S] < E[1/S]$ or $\sigma/\sqrt{ES^2} < \sigma E[1/S]$, that is, $P = \frac{\sigma}{\sqrt{ES^2}} < E\left[\frac{\sigma}{S}\right] = E\left[\frac{\hat{C}p}{Cp}\right]$. Therefore $P = (1-\bar{\rho})^{-1/2}$ is a lower bound on $E\left[\hat{C}p/Cp\right]$.

Suppose the process follows an AR(1), that is, $(X_t - \mu) - \phi(X_{t-1} - \mu) = a_t$, where the a_t are *iid* $N(0, \sigma_a^2)$ and $-1 < \phi < 1$. It is well known that the variance of the process characteristic X_t is given by $Var[X_t] = \sigma^2 = \sigma_a^2 / (1-\phi^2)$. Also, the covariance j periods apart is given by $Cov[X_t, X_{t-j}] = \phi^j Var[X_t] = \phi^j \sigma^2 = \phi^j \sigma_a^2 / (1-\phi^2)$ and the correlation j periods apart is given by $Corr[X_t, X_{t-j}] = \phi^j$.

Consider a sample $\{X_t\}_n$ from an AR(1) process, consecutive in time, taken at the uniform time interval consistent with the parameter ϕ. Let $S^2 = \frac{1}{n-1}\sum_{t=1}^{n}(X_t - \bar{X})^2$ be the

Table 4.1. Values of $(1-\bar{\rho}) = ES^2/\sigma^2$ for AR(1)

ϕ	$n=10$	$n=20$	$n=30$	$n=40$	$n=50$
0.1	0.97805	0.98895	0.99262	0.99446	0.99557
0.2	0.95139	0.97532	0.98348	0.98758	0.99005
0.3	0.91837	0.95811	0.97185	0.97881	0.98301
0.4	0.87654	0.93567	0.95658	0.96723	0.97370
0.5	0.82218	0.90526	0.93563	0.95128	0.96082
0.6	0.74950	0.86184	0.90517	0.92788	0.94184
0.7	0.64943	0.79529	0.85696	0.89031	0.91111
0.8	0.50783	0.68300	0.77006	0.82051	0.85306
0.9	0.30264	0.46873	0.57744	0.65214	0.70574

sample variance. This is commonly called the unbiased estimator of σ^2, but of course, it is no longer so. Its relative bias will depend on the parameter ϕ and the sample size n.

Table 4.1 shows, for selected sample size n and parameter ϕ, the estimation bias ratio $(1-\bar{\rho})$ for the AR(1) model. See Appendix C for the computation of $\bar{\rho}$. Note that since $(1-\bar{\rho}) = ES^2/\sigma^2$, this bias ratio gives the mean of the sampling distribution of S^2 as a proportion of the true variance of X. From Table 4.1 we see, for example, that for $\phi = 0.9$ and $n = 20$, we would be underestimating the true process variance by (1 - 0.46873) or about 53 percent on average if we use $S^2 = \frac{1}{n-1}\sum_{t=1}^{n}(X_t - \bar{X})^2$ as an estimator of σ^2 in an AR(1) process.

Table 4.2 gives the value of $\mathrm{P} = (1-\bar{\rho})^{-1/2}$ for selected sample size n and parameter ϕ in the AR(1) model. This value provides a lower bound on $E\left[\hat{C}p/Cp\right]$. From Table 4.2 we see, for example, that for $\phi = 0.9$ and $n = 20$, we would be overestimating the true process Cp by at least 46 percent on average if we use $S^2 = \frac{1}{n-1}\sum_{t=1}^{n}(X_t - \bar{X})^2$ as an estimate of σ^2 in an AR(1) process.

Table 4.2. Values of $P = (1-\bar{\rho})^{-1/2} < E[\hat{C}p/Cp]$ for AR(1)

ϕ	$n = 10$	$n = 20$	$n = 30$	$n = 40$	$n = 50$
0.1	1.01116	1.00557	1.00371	1.00278	1.00223
0.2	1.02523	1.01257	1.00837	1.00627	1.00501
0.3	1.04350	1.02163	1.01438	1.01077	1.00861
0.4	1.06811	1.03380	1.02245	1.01680	1.01342
0.5	1.10285	1.05102	1.03383	1.02529	1.02019
0.6	1.15509	1.07718	1.05108	1.03813	1.03042
0.7	1.24088	1.12134	1.08024	1.05981	1.04765
0.8	1.40326	1.21001	1.13956	1.10397	1.08271
0.9	1.81775	1.46063	1.31598	1.23831	1.19036

Now consider the AR(2) process $(X_t - \mu) - \phi_1(X_{t-1} - \mu) - \phi_2(X_{t-2} - \mu) = a_t$, where again a_t are *iid* $N(0, \sigma_a^2)$. The requirement of stationarity restricts the parameters (ϕ_1, ϕ_2) to the joint conditions $\phi_2 + \phi_1 < 1$, $\phi_2 - \phi_1 < 1$, and $-2 < \phi_1 < 2$.

Tables 4.3 through 4.7 gives the lower bound $P = (1-\bar{\rho})^{-1/2}$ of $E[\hat{C}p/Cp]$ for an AR(2) process, for values of (ϕ_1, ϕ_2) and n. An asterisk indicates parameter values outside the region of stationarity. We see that P can be quite high for (ϕ_1, ϕ_2) in certain regions of the parameter space. For example, when (ϕ_1, ϕ_2) = (0.7, 0.2) and n = 10, we have P = 1.86888, indicating that in repeated samples we are overestimating the true process Cp by at least 86 percent. On the other hand, when (ϕ_1, ϕ_2) = (0.7, 0.2) and n = 50, we have P = 1.22473, indicating that in repeated samples we are overestimating the true process Cp by at least 23 percent. The damage, in terms of the relative bias of the estimator, is "contained" somewhat by the larger sample size n.

4.5. Estimating *Cp* when μ is Known and σ is Unknown

Let $\{X_i\}_n$ be multivariate normal with equal means μ, equal variances σ^2, and correlations $\rho(i, j)$ not necessarily zero. In this section we assume a known process mean μ

Table 4.3. Values of $P = (1-\bar{\rho})^{-1/2} < E\left[\hat{C}p/Cp\right]$ for AR(2) with $n = 10$

$\phi_1 \backslash \phi_2$	−0.9	−0.8	−0.6	−0.4	−0.2	0.0	0.2	0.4	0.6	0.8	0.9
-1.8	0.95266	*	*	*	*	*	*	*	*	*	*
-1.7	0.95164	0.95210	*	*	*	*	*	*	*	*	*
-1.6	0.95029	0.95174	*	*	*	*	*	*	*	*	*
-1.5	0.95012	0.95108	0.95158	*	*	*	*	*	*	*	*
-1.4	0.95105	0.95119	0.95215	*	*	*	*	*	*	*	*
-1.3	0.95245	0.95199	0.95239	0.95153	*	*	*	*	*	*	*
-1.2	0.95362	0.95301	0.95289	0.95282	*	*	*	*	*	*	*
-1.1	0.95416	0.95383	0.95366	0.95376	0.95187	*	*	*	*	*	*
-1.0	0.95398	0.95422	0.95450	0.95478	0.95397	*	*	*	*	*	*
-0.9	0.95325	0.95419	0.95529	0.95595	0.95581	0.95275	*	*	*	*	*
-0.8	0.95235	0.95393	0.95597	0.95723	0.95773	0.95613	*	*	*	*	*
-0.7	0.95165	0.95371	0.95663	0.95861	0.95984	0.95947	0.95467	*	*	*	*
-0.6	0.95147	0.95377	0.95736	0.96010	0.96220	0.96307	0.96053	*	*	*	*
-0.5	0.95196	0.95427	0.95826	0.96175	0.96484	0.96712	0.96688	0.95916	*	*	*
-0.4	0.95310	0.95522	0.95940	0.96360	0.96783	0.97176	0.97413	0.97086	*	*	*
-0.3	0.95469	0.95652	0.96078	0.96571	0.97122	0.97714	0.98268	0.98477	0.97209	*	*
-0.2	0.95642	0.95797	0.96235	0.96810	0.97511	0.98345	0.99296	1.00202	1.00228	*	*
-0.1	0.95792	0.95933	0.96404	0.97081	0.97962	0.99094	1.00560	1.02416	1.04363	1.03369	*
0.0	0.95886	0.96039	0.96579	0.97389	0.98492	1.00000	1.02151	1.05367	1.10406	1.18784	1.25192
0.1	0.95908	0.96103	0.96760	0.97743	0.99121	1.01116	1.04210	1.09474	1.20035	1.56697	*
0.2	0.95856	0.96131	0.96957	0.98157	0.99882	1.02523	1.06969	1.15533	1.37673	*	*
0.3	0.95756	0.96142	0.97185	0.98652	1.00819	1.04350	1.10834	1.25258	1.81267	*	*
0.4	0.95650	0.96172	0.97467	0.99254	1.02003	1.06810	1.16583	1.43196	*	*	*
0.5	0.95594	0.96261	0.97829	1.00002	1.03543	1.10285	1.25896	1.87831	*	*	*
0.6	0.95644	0.96449	0.98294	1.00944	1.05626	1.15509	1.43252	*	*	*	*
0.7	0.95842	0.96760	0.98879	1.02163	1.08593	1.24088	1.86888	*	*	*	*
0.8	0.96202	0.97192	0.99598	1.03801	1.13120	1.40326	*	*	*	*	*
0.9	0.96693	0.97712	1.00476	1.06146	1.20712	1.81775	*	*	*	*	*
1.0	0.97237	0.98255	1.01591	1.09801	1.35427	*	*	*	*	*	*
1.1	0.97706	0.98752	1.03159	1.16154	1.73850	*	*	*	*	*	*
1.2	0.97963	0.99187	1.05711	1.28984	*	*	*	*	*	*	*
1.3	0.97942	0.99709	1.10535	1.63711	*	*	*	*	*	*	*
1.4	0.97780	1.00824	1.21132	*	*	*	*	*	*	*	*
1.5	0.98013	1.03783	1.51624	*	*	*	*	*	*	*	*
1.6	0.99920	1.11854	*	*	*	*	*	*	*	*	*
1.7	1.06669	1.37768	*	*	*	*	*	*	*	*	*
1.8	1.30266	*	*	*	*	*	*	*	*	*	*

Table 4.4. Values of $P = (1-\bar{\rho})^{-1/2} < E\left[\hat{C}p/Cp\right]$ for AR(2) with $n = 20$

$\phi_1 \backslash \phi_2$	−0.9	−0.8	−0.6	−0.4	−0.2	0.0	0.2	0.4	0.6	0.8	0.9
-1.8	0.97512	*	*	*	*	*	*	*	*	*	*
-1.7	0.97561	0.97532	*	*	*	*	*	*	*	*	*
-1.6	0.97539	0.97555	*	*	*	*	*	*	*	*	*
-1.5	0.97531	0.97556	0.97551	*	*	*	*	*	*	*	*
-1.4	0.97572	0.97565	0.97574	*	*	*	*	*	*	*	*
-1.3	0.97586	0.97589	0.97598	0.97569	*	*	*	*	*	*	*
-1.2	0.97561	0.97600	0.97624	0.97613	*	*	*	*	*	*	*
-1.1	0.97553	0.97603	0.97653	0.97661	0.97597	*	*	*	*	*	*
-1.0	0.97585	0.97617	0.97683	0.97712	0.97680	*	*	*	*	*	*
-0.9	0.97623	0.97643	0.97716	0.97768	0.97767	0.97650	*	*	*	*	*
-0.8	0.97628	0.97666	0.97752	0.97828	0.97862	0.97800	*	*	*	*	*
-0.7	0.97607	0.97676	0.97791	0.97895	0.97967	0.97961	0.97758	*	*	*	*
-0.6	0.97595	0.97686	0.97833	0.97969	0.98084	0.98141	0.98041	*	*	*	*
-0.5	0.97621	0.97709	0.97879	0.98050	0.98216	0.98346	0.98359	0.98010	*	*	*
-0.4	0.97669	0.97743	0.97930	0.98140	0.98363	0.98581	0.98731	0.98616	*	*	*
-0.3	0.97704	0.97777	0.97988	0.98241	0.98531	0.98852	0.99171	0.99350	0.98782	*	*
-0.2	0.97701	0.97801	0.98050	0.98355	0.98723	0.99170	0.99701	1.00270	1.00497	*	*
-0.1	0.97678	0.97820	0.98120	0.98484	0.98946	0.99546	1.00353	1.01460	1.02901	1.03195	*
0.0	0.97676	0.97848	0.98199	0.98632	0.99205	1.00000	1.01172	1.03059	1.06519	1.14193	1.22199
0.1	0.97720	0.97896	0.98289	0.98803	0.99514	1.00557	1.02232	1.05315	1.12521	1.43284	*
0.2	0.97791	0.97956	0.98393	0.99003	0.99884	1.01257	1.03656	1.08724	1.24153	*	*
0.3	0.97841	0.98010	0.98511	0.99240	1.00340	1.02163	1.05669	1.14409	1.55119	*	*
0.4	0.97841	0.98053	0.98649	0.99525	1.00911	1.03380	1.08719	1.25525	*	*	*
0.5	0.97817	0.98102	0.98814	0.99875	1.01651	1.05102	1.13838	1.55527	*	*	*
0.6	0.97834	0.98181	0.99013	1.00314	1.02645	1.07718	1.23970	*	*	*	*
0.7	0.97928	0.98296	0.99256	1.00882	1.04051	1.12134	1.51859	*	*	*	*
0.8	0.98057	0.98421	0.99558	1.01644	1.06188	1.21002	*	*	*	*	*
0.9	0.98131	0.98537	0.99950	1.02721	1.09815	1.46063	*	*	*	*	*
1.0	0.98122	0.98673	1.00480	1.04357	1.17204	*	*	*	*	*	*
1.1	0.98141	0.98899	1.01224	1.07139	1.38836	*	*	*	*	*	*
1.2	0.98341	0.99243	1.02347	1.12869	*	*	*	*	*	*	*
1.3	0.98676	0.99650	1.04267	1.30439	*	*	*	*	*	*	*
1.4	0.98874	1.00183	1.08210	*	*	*	*	*	*	*	*
1.5	0.98971	1.01293	1.20922	*	*	*	*	*	*	*	*
1.6	0.99765	1.03541	*	*	*	*	*	*	*	*	*
1.7	1.01453	1.10153	*	*	*	*	*	*	*	*	*
1.8	1.04147	*	*	*	*	*	*	*	*	*	*

Table 4.5. Values of $P = (1-\bar{\rho})^{-1/2} < E\left[\hat{C}p/Cp\right]$ for AR(2) with $n = 30$

$\phi_1 \backslash \phi_2$	−0.9	−0.8	−0.6	−0.4	−0.2	0.0	0.2	0.4	0.6	0.8	0.9
-1.8	0.98355	*	*	*	*	*	*	*	*	*	*
-1.7	0.98350	0.98354	*	*	*	*	*	*	*	*	*
-1.6	0.98358	0.98358	*	*	*	*	*	*	*	*	*
-1.5	0.98356	0.98366	0.98361	*	*	*	*	*	*	*	*
-1.4	0.98370	0.98372	0.98376	*	*	*	*	*	*	*	*
-1.3	0.98363	0.98381	0.98392	0.98373	*	*	*	*	*	*	*
-1.2	0.98371	0.98387	0.98409	0.98402	*	*	*	*	*	*	*
-1.1	0.98384	0.98398	0.98427	0.98434	0.98394	*	*	*	*	*	*
-1.0	0.98377	0.98407	0.98447	0.98467	0.98447	*	*	*	*	*	*
-0.9	0.98382	0.98415	0.98469	0.98504	0.98505	0.98432	*	*	*	*	*
-0.8	0.98400	0.98427	0.98492	0.98545	0.98568	0.98529	*	*	*	*	*
-0.7	0.98402	0.98440	0.98517	0.98589	0.98639	0.98636	0.98506	*	*	*	*
-0.6	0.98397	0.98450	0.98545	0.98637	0.98717	0.98757	0.98694	*	*	*	*
-0.5	0.98412	0.98464	0.98575	0.98691	0.98804	0.98895	0.98908	0.98681	*	*	*
-0.4	0.98430	0.98481	0.98608	0.98750	0.98902	0.99052	0.99158	0.99090	*	*	*
-0.3	0.98429	0.98497	0.98645	0.98817	0.99014	0.99234	0.99455	0.99588	0.99224	*	*
-0.2	0.98428	0.98512	0.98686	0.98892	0.99142	0.99446	0.99812	1.00214	1.00411	*	*
-0.1	0.98449	0.98534	0.98732	0.98977	0.99289	0.99697	1.00250	1.01023	1.02082	1.02549	*
0.0	0.98471	0.98558	0.98783	0.99075	0.99462	1.00000	1.00800	1.02108	1.04611	1.10916	1.18853
0.1	0.98472	0.98580	0.98842	0.99187	0.99666	1.00371	1.01510	1.03641	1.08863	1.34217	*
0.2	0.98474	0.98607	0.98909	0.99319	0.99911	1.00837	1.02464	1.05963	1.17338	*	*
0.3	0.98505	0.98641	0.98986	0.99474	1.00212	1.01438	1.03809	1.09876	1.41126	*	*
0.4	0.98535	0.98678	0.99076	0.99661	1.00590	1.02244	1.05850	1.17728	*	*	*
0.5	0.98538	0.98716	0.99182	0.99891	1.01077	1.03383	1.09294	1.40110	*	*	*
0.6	0.98550	0.98765	0.99311	1.00178	1.01730	1.05108	1.16257	*	*	*	*
0.7	0.98601	0.98827	0.99468	1.00549	1.02650	1.08024	1.36497	*	*	*	*
0.8	0.98642	0.98893	0.99664	1.01045	1.04045	1.13956	*	*	*	*	*
0.9	0.98650	0.98974	0.99917	1.01744	1.06403	1.31598	*	*	*	*	*
1.0	0.98702	0.99086	1.00256	1.02802	1.11219	*	*	*	*	*	*
1.1	0.98801	0.99220	1.00733	1.04589	1.25889	*	*	*	*	*	*
1.2	0.98843	0.99395	1.01453	1.08241	*	*	*	*	*	*	*
1.3	0.98931	0.99663	1.02667	1.19600	*	*	*	*	*	*	*
1.4	0.99145	1.00036	1.05141	*	*	*	*	*	*	*	*
1.5	0.99279	1.00695	1.12903	*	*	*	*	*	*	*	*
1.6	0.99725	1.01984	*	*	*	*	*	*	*	*	*
1.7	1.00361	1.06109	*	*	*	*	*	*	*	*	*
1.8	1.02960	*	*	*	*	*	*	*	*	*	*

Table 4.6. Values of $P = (1-\bar{\rho})^{-1/2} < E\left[\hat{C}p/Cp\right]$ for AR(2) with n = 40

$\phi_1 \backslash \phi_2$	−0.9	−0.8	−0.6	−0.4	−0.2	0.0	0.2	0.4	0.6	0.8	0.9
-1.8	0.98758	*	*	*	*	*	*	*	*	*	*
-1.7	0.98760	0.98762	*	*	*	*	*	*	*	*	*
-1.6	0.98766	0.98766	*	*	*	*	*	*	*	*	*
-1.5	0.98766	0.98772	0.98768	*	*	*	*	*	*	*	*
-1.4	0.98770	0.98777	0.98779	*	*	*	*	*	*	*	*
-1.3	0.98772	0.98782	0.98791	0.98778	*	*	*	*	*	*	*
-1.2	0.98777	0.98788	0.98804	0.98799	*	*	*	*	*	*	*
-1.1	0.98777	0.98794	0.98817	0.98823	0.98794	*	*	*	*	*	*
-1.0	0.98785	0.98801	0.98832	0.98848	0.98833	*	*	*	*	*	*
-0.9	0.98785	0.98808	0.98848	0.98875	0.98876	0.98822	*	*	*	*	*
-0.8	0.98790	0.98816	0.98865	0.98905	0.98924	0.98895	*	*	*	*	*
-0.7	0.98797	0.98825	0.98884	0.98938	0.98977	0.98976	0.98879	*	*	*	*
-0.6	0.98797	0.98834	0.98905	0.98975	0.99035	0.99067	0.99020	*	*	*	*
-0.5	0.98805	0.98844	0.98927	0.99015	0.99101	0.99170	0.99181	0.99013	*	*	*
-0.4	0.98812	0.98855	0.98952	0.99059	0.99174	0.99288	0.99370	0.99321	*	*	*
-0.3	0.98813	0.98866	0.98979	0.99109	0.99258	0.99425	0.99594	0.99699	0.99431	*	*
-0.2	0.98824	0.98880	0.99009	0.99165	0.99354	0.99584	0.99863	1.00173	1.00336	*	*
-0.1	0.98832	0.98894	0.99043	0.99228	0.99464	0.99773	1.00193	1.00785	1.01612	1.02070	*
0.0	0.98834	0.98910	0.99081	0.99301	0.99593	1.00000	1.00606	1.01605	1.03545	1.08749	1.16156
0.1	0.98849	0.98928	0.99125	0.99385	0.99746	1.00278	1.01140	1.02762	1.06805	1.28021	*
0.2	0.98859	0.98948	0.99174	0.99483	0.99929	1.00627	1.01856	1.04514	1.13386	*	*
0.3	0.98864	0.98971	0.99231	0.99599	1.00154	1.01077	1.02865	1.07470	1.32527	*	*
0.4	0.98884	0.98998	0.99298	0.99738	1.00435	1.01680	1.04392	1.13462	*	*	*
0.5	0.98897	0.99028	0.99377	0.99909	1.00799	1.02529	1.06970	1.31131	*	*	*
0.6	0.98907	0.99063	0.99472	1.00122	1.01285	1.03813	1.12211	*	*	*	*
0.7	0.98937	0.99105	0.99587	1.00397	1.01970	1.05981	1.27910	*	*	*	*
0.8	0.98954	0.99154	0.99733	1.00766	1.03004	1.10397	*	*	*	*	*
0.9	0.98979	0.99217	0.99920	1.01284	1.04748	1.23831	*	*	*	*	*
1.0	0.99025	0.99293	1.00170	1.02066	1.08302	*	*	*	*	*	*
1.1	0.99051	0.99392	1.00521	1.03381	1.19255	*	*	*	*	*	*
1.2	0.99122	0.99524	1.01050	1.06059	*	*	*	*	*	*	*
1.3	0.99182	0.99709	1.01939	1.14373	*	*	*	*	*	*	*
1.4	0.99296	0.99987	1.03742	*	*	*	*	*	*	*	*
1.5	0.99435	1.00456	1.09334	*	*	*	*	*	*	*	*
1.6	0.99725	1.01394	*	*	*	*	*	*	*	*	*
1.7	1.00196	1.04288	*	*	*	*	*	*	*	*	*
1.8	1.01731	*	*	*	*	*	*	*	*	*	*

Table 4.7. Values of $P = (1-\bar{\rho})^{-1/2} < E\left[\hat{C}p/Cp\right]$ for AR(2) with $n = 50$

$\phi_1 \backslash \phi_2$	−0.9	−0.8	−0.6	−0.4	−0.2	0.0	0.2	0.4	0.6	0.8	0.9
-1.8	0.99007	*	*	*	*	*	*	*	*	*	*
-1.7	0.99009	0.99008	*	*	*	*	*	*	*	*	*
-1.6	0.99011	0.99012	*	*	*	*	*	*	*	*	*
-1.5	0.99012	0.99016	0.99013	*	*	*	*	*	*	*	*
-1.4	0.99014	0.99020	0.99022	*	*	*	*	*	*	*	*
-1.3	0.99017	0.99024	0.99031	0.99021	*	*	*	*	*	*	*
-1.2	0.99018	0.99029	0.99042	0.99038	*	*	*	*	*	*	*
-1.1	0.99022	0.99034	0.99052	0.99057	0.99034	*	*	*	*	*	*
-1.0	0.99023	0.99039	0.99064	0.99077	0.99065	*	*	*	*	*	*
-0.9	0.99028	0.99045	0.99077	0.99099	0.99100	0.99057	*	*	*	*	*
-0.8	0.99029	0.99051	0.99090	0.99123	0.99138	0.99115	*	*	*	*	*
-0.7	0.99035	0.99058	0.99105	0.99149	0.99180	0.99180	0.99103	*	*	*	*
-0.6	0.99036	0.99065	0.99122	0.99178	0.99227	0.99253	0.99215	*	*	*	*
-0.5	0.99042	0.99073	0.99139	0.99210	0.99279	0.99335	0.99345	0.99211	*	*	*
-0.4	0.99045	0.99081	0.99159	0.99245	0.99338	0.99430	0.99497	0.99459	*	*	*
-0.3	0.99050	0.99090	0.99181	0.99285	0.99405	0.99539	0.99677	0.99763	0.99550	*	*
-0.2	0.99056	0.99101	0.99205	0.99330	0.99481	0.99667	0.99892	1.00144	1.00282	*	*
-0.1	0.99060	0.99112	0.99232	0.99380	0.99570	0.99818	1.00157	1.00636	1.01314	1.01731	*
0.0	0.99068	0.99125	0.99262	0.99438	0.99673	1.00000	1.00488	1.01295	1.02875	1.07257	1.14031
0.1	0.99072	0.99139	0.99296	0.99505	0.99795	1.00223	1.00916	1.02223	1.05508	1.23585	*
0.2	0.99082	0.99155	0.99336	0.99583	0.99941	1.00501	1.01488	1.03627	1.10851	*	*
0.3	0.99090	0.99173	0.99381	0.99676	1.00121	1.00861	1.02295	1.05996	1.26753	*	*
0.4	0.99099	0.99193	0.99434	0.99787	1.00345	1.01342	1.03514	1.10810	*	*	*
0.5	0.99112	0.99217	0.99497	0.99922	1.00635	1.02019	1.05569	1.25297	*	*	*
0.6	0.99122	0.99245	0.99572	1.00092	1.01022	1.03042	1.09751	*	*	*	*
0.7	0.99142	0.99278	0.99664	1.00311	1.01567	1.04764	1.22473	*	*	*	*
0.8	0.99155	0.99317	0.99779	1.00604	1.02389	1.08271	*	*	*	*	*
0.9	0.99181	0.99365	0.99927	1.01015	1.03772	1.19035	*	*	*	*	*
1.0	0.99202	0.99425	1.00125	1.01636	1.06584	*	*	*	*	*	*
1.1	0.99241	0.99503	1.00403	1.02677	1.15270	*	*	*	*	*	*
1.2	0.99276	0.99606	1.00821	1.04790	*	*	*	*	*	*	*
1.3	0.99341	0.99751	1.01523	1.11326	*	*	*	*	*	*	*
1.4	0.99412	0.99969	1.02941	*	*	*	*	*	*	*	*
1.5	0.99532	1.00335	1.07312	*	*	*	*	*	*	*	*
1.6	0.99741	1.01070	*	*	*	*	*	*	*	*	*
1.7	1.00142	1.03315	*	*	*	*	*	*	*	*	*
1.8	1.01370	*	*	*	*	*	*	*	*	*	*

and so we use $S = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2}$ in the definition of $\hat{C}p$. The lower bound for $E\left[\frac{\sigma}{S}\right] = E\left[\frac{\hat{C}p}{Cp}\right]$ in this case is not dramatic. Since

$$ES^2 = E\left[\frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2\right] = \frac{1}{n}\left\{\sum_{i=1}^{n}E\left[(X_i - \mu)^2\right]\right\} = \frac{1}{n}n\sigma^2 = \sigma^2, \tag{4.41}$$

we have

$$\frac{\sigma^2}{ES^2} = 1 = \frac{\sigma}{\sqrt{ES^2}} < E\left[\frac{\sigma}{S}\right] = E\left[\frac{\hat{C}p}{Cp}\right],$$

and so the lower bound becomes 1. That is,

$$1 < E\left[\frac{\sigma}{S}\right] = E\left[\frac{\hat{C}p}{Cp}\right]. \tag{4.42}$$

To sharpen this lower bound, we will derive a Taylor series approximation to $E\left[\hat{C}p/Cp\right]$ in this very important case where the sampled data is subject to autocorrelation.

Recall the Taylor series approximation,

$$E[g(Y)] \approx gE[Y] + \frac{1}{2}g''E[Y]\cdot Var[Y], \tag{4.43}$$

where we let $Y = \sum_{i=1}^{n}(X_i - \mu)^2$. We will need both $E\left[\sum_{i=1}^{n}(X_i - \mu)^2\right]$ and $Var\left[\sum_{i=1}^{n}(X_i - \mu)^2\right]$.

Now

$$E\left[\sum_{i=1}^{n}(X_i - \mu)^2\right] = \sum_{i=1}^{n}E(X_i - \mu)^2 = \sum_{i=1}^{n}\sigma^2 = n\sigma^2, \tag{4.44}$$

but the variance is more difficult. We begin with

$$Var\left[\sum_{i=1}^{n}(X_i - \mu)^2\right] = E\left[\left\{\sum_{i=1}^{n}(X_i - \mu)^2\right\}^2\right] - E^2\left[\sum_{i=1}^{n}(X_i - \mu)^2\right]. \tag{4.45}$$

We know the second term on the right straightaway as

$$E^2\left[\sum_{i=1}^{n}(X_i-\mu)^2\right]=\left(n\sigma^2\right)^2=n^2\sigma^4, \tag{4.46}$$

but the mean of the square is more difficult. Continuing, we have

$$\begin{aligned}
E\left[\left\{\sum_{i=1}^{n}(X_i-\mu)^2\right\}^2\right] &= E\left[\left\{(X_1-\mu)^2+(X_2-\mu)^2+\cdots+(X_n-\mu)^2\right\}^2\right]\\
&= E\left[\sum_{i=1}^{n}(X_i-\mu)^4+\sum_{i\neq j}^{n(n-1)}(X_i-\mu)^2(X_j-\mu)^2\right]\\
&= E\left[\sum_{i=1}^{n}(X_i-\mu)^4\right]+E\left[\sum_{i\neq j}^{n(n-1)}(X_i-\mu)^2(X_j-\mu)^2\right]\\
&= \sum_{i=1}^{n}E\left[(X_i-\mu)^4\right]+\sum_{i\neq j}^{n(n-1)}E\left[(X_i-\mu)^2(X_j-\mu)^2\right].
\end{aligned} \tag{4.47}$$

Consider the ith term in the first summation of terms on the right of equation (4.47). We have

$$\begin{aligned}
(X_i-\mu)^4 &= (X_i-\mu)^2(X_i-\mu)^2\\
&= \left(X_i-2\mu X_i+\mu^2\right)\left(X_i-2\mu X_i+\mu^2\right)\\
&= X_i^4-4\mu X_i^3+6\mu^2X_i^2-4\mu^3X_i+\mu^4.
\end{aligned} \tag{4.48}$$

Taking expectations and moments from Appendix B gives

$$\begin{aligned}
E\left[(X_i-\mu)^4\right] &= E\left[X_i^4-4\mu X_i^3+6\mu^2X_i^2-4\mu^3X_i+\mu^4\right]\\
&= E\left[X_i^4\right]-4\mu E\left[X_i^3\right]+6\mu^2E\left[X_i^2\right]-4\mu^3E\left[X_i\right]+\mu^4\\
&= 3\sigma^4+6\mu^2\sigma^2+\mu^4-4\mu\left(3\mu\sigma^2+\mu^3\right)+6\mu^2\left(\sigma^2+\mu^2\right)-4\mu^3(\mu)+\mu^4\\
&= 3\sigma^4+6\mu^2\sigma^2+\mu^4-6\mu^2\sigma^2-2\mu^4+\mu^2\sigma^2+\mu^4\\
&\quad -6\mu^2\sigma^2-2\mu^4+4\mu^2\sigma^2+4\mu^4-2\mu^4+\mu^2\sigma^2+\mu^4-2\mu^4+\mu^4\\
&= 3\sigma^4(1)+\mu^2\sigma^2(6-6+1-6+4+1)+\mu^4(1-2+1-2+4-2+1-2+1)\\
&= 3\sigma^4(1)+\mu^2\sigma^2(0)+\mu^4(0)
\end{aligned}$$

$$= 3\sigma^4, \tag{4.49}$$

and so

$$\sum_{i=1}^{n} E\left[(X_i - \mu)^4\right] = \sum_{i=1}^{n} 3\sigma^4 = 3n\sigma^4. \tag{4.50}$$

The second summation on the right side of equation (4.47) contains cross products and is more difficult. Taking the (i, j)th term, we have

$$\begin{aligned}(X_i - \mu)^2 (X_j - \mu)^2 &= \left(X_i^2 - 2\mu X_i + \mu^2\right)\left(X_j^2 - 2\mu X_j + \mu^2\right) \\ &= X_i^2 X_j^2 - 2\mu X_i^2 X_j - 2\mu X_i X_j^2 + \mu^2 X_i^2 + \mu^2 X_j^2 \\ &\quad + 4\mu^2 X_i X_j - 2\mu^3 X_i - 2\mu^3 X_j + \mu^4. \end{aligned} \tag{4.51}$$

Taking expectations gives

$$\begin{aligned} E\left[(X_i - \mu)^2 (X_j - \mu)^2\right] &= E\left[X_i^2 X_j^2\right] - 2\mu E\left[X_i^2 X_j\right] - 2\mu E\left[X_i X_j^2\right] \\ &+ \mu^2 E\left[X_i^2\right] + \mu^2 E\left[X_j^2\right] + 4\mu^2 E\left[X_i X_j\right] - 2\mu^3 E\left[X_i\right] - 2\mu^3 E\left[X_j\right] + \mu^4. \end{aligned} \tag{4.52}$$

Taking moments from Appendix B gives

$$\begin{aligned} E\left[(X_i - \mu)^2 (X_j - \mu)^2\right] &= \sigma^4\left(1 + 2\rho_{ij}^2\right) + 2\mu^2\sigma^2\left(1 + 2\rho_{ij}\right) + \mu^4 \\ &- 2\mu\left(\mu\sigma^2\left(1 + 2\rho_{ij}\right) + \mu^3\right) - 2\mu\left(\mu\sigma^2\left(1 + 2\rho_{ij}\right) + \mu^3\right) + \mu^2\left(\sigma^2 + \mu^2\right) \\ &+ \mu^2\left(\sigma^2 + \mu^2\right) + 4\mu^2\left(\sigma^2\rho_{ij} + \mu^2\right) - 2\mu^3\mu - 2\mu^3\mu + \mu^4, \end{aligned}$$

or

$$\begin{aligned} E\left[(X_i - \mu)^2 (X_j - \mu)^2\right] &= \sigma^4 + 2\sigma^4\rho_{ij}^2 + 2\mu^2\sigma^2 + 4\mu^2\sigma^2\rho_{ij} + \mu^4 \\ &- 2\mu^2\sigma^2 - 4\mu^2\sigma^2\rho_{ij} - 2\mu^4 - 2\mu^2\sigma^2 - 4\mu^2\sigma^2\rho_{ij} - 2\mu^4 + \mu^2\sigma^2 + \mu^4 \\ &+ \mu^2\sigma^2 + \mu^4 + 4\mu^2\sigma^2\rho_{ij} + 4\mu^4 - 2\mu^4 - 2\mu^4 + \mu^4, \end{aligned}$$

or

$$E\left[(X_i-\mu)^2(X_j-\mu)^2\right]=\sigma^4(1)+\sigma^4\rho_{ij}^2(2)$$

$$+\mu^2\sigma^2(2-2-2+1+1)+\mu^2\sigma^2\rho_{ij}(4-4-4+4)$$

$$+\mu^4(1-2-2+1+1+4-2-2+1),$$

or

$$E\left[(X_i-\mu)^2(X_j-\mu)^2\right]=\sigma^4(1)+\sigma^4\rho_{ij}^2(2)+\mu^2\sigma^2(0)+\mu^2\sigma^2\rho_{ij}(0)+\mu^4(0)$$

$$=\sigma^4\left(1+2\rho_{ij}^2\right). \tag{4.53}$$

We have

$$\sum_{i\neq j}^{n(n-1)}E\left[(X_i-\mu)^2(X_j-\mu)^2\right]=\sum_{i\neq j}^{n(n-1)}\sigma^4\left(1+2\rho_{ij}^2\right)$$

$$=\sigma^4\sum_{i\neq j}^{n(n-1)}\left(1+2\rho_{ij}^2\right). \tag{4.54}$$

Therefore,

$$Var\left[\sum_{i=1}^{n}(X_i-\mu)^2\right]=3n\sigma^4+\sigma^4\sum_{i\neq j}^{n(n-1)}\left(1+2\rho_{ij}^2\right)-n^2\sigma^4$$

$$=3n\sigma^4+n(n-1)\sigma^4+2\sigma^4\sum_{i\neq j}^{n(n-1)}\rho_{ij}^2-n^2\sigma^4$$

$$=3n\sigma^4+n^2\sigma^4-n\sigma^4+2\sigma^4\sum_{i\neq j}^{n(n-1)}\rho_{ij}^2-n^2\sigma^4$$

$$=2n\sigma^4+2\sigma^4\sum_{i\neq j}^{n(n-1)}\rho_{ij}^2$$

$$=2\sigma^4\left\{n+\sum_{i\neq j}^{n(n-1)}\rho_{ij}^2\right\}$$

$$=2\sigma^4\left\{n+n(n-1)\overline{\rho}^2\right\}$$

$$=2n\sigma^4\left\{1+(n-1)\overline{\rho}^2\right\}, \tag{4.55}$$

where $\bar{\rho}^2 = \frac{1}{n(n-1)} \sum_{i \neq j}^{n(n-1)} \rho_{ij}^2$ is the *average of the squares* of the $n(n-1)$ pairwise correlations.

We emphasize that $\bar{\rho}^2$ is the average of the squares and not the square of the average, as our unfortunate symbol might suggest. Collecting results, we have

$$E\left[\sum_{i=1}^{n}(X_i-\mu)^2\right] = n\sigma^2 \tag{4.56}$$

and

$$Var\left[\sum_{i=1}^{n}(X_i-\mu)^2\right] = 2n\sigma^4\left\{1+(n-1)\bar{\rho}^2\right\}. \tag{4.57}$$

For completeness, we note

$$E\left[S^2\right] = E\left[\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2\right] = \frac{1}{n}E\left[\sum_{i=1}^{n}(X_i-\mu)^2\right] = \frac{1}{n}n\sigma^2 = \sigma^2 \tag{4.58}$$

and

$$\begin{aligned} Var\left[S^2\right] &= Var\left[\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2\right] = \frac{1}{n^2}Var\left[\sum_{i=1}^{n}(X_i-\mu)^2\right] \\ &= \frac{1}{n^2}2n\sigma^4\left\{1+(n-1)\bar{\rho}^2\right\} = 2\sigma^4\left\{\frac{1}{n}+\left\{1-\frac{1}{n}\right\}\bar{\rho}^2\right\}, \end{aligned} \tag{4.59}$$

where $S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2$.

To get an approximation for $E\left[\hat{C}p/Cp\right]$, first take $Y = \sum_{i=1}^{n}(X_i-\mu)^2$. We know that $E[Y] = n\sigma^2$ and $Var[Y] = 2n\sigma^4\left\{1+(n-1)\bar{\rho}^2\right\}$. Letting $g(Y) = \frac{1}{\sqrt{Y}} = Y^{-1/2}$ gives $g'(Y) = -\frac{1}{2}Y^{-3/2}$ and $g''(Y) = \frac{3}{4}Y^{-5/2}$. We have

$$\begin{aligned} E[g(Y)] &\approx gE[Y] + \frac{1}{2}g''E[Y]\cdot Var[Y] \\ &= \left(n\sigma^2\right)^{-1/2} + \frac{1}{2}\frac{3}{4}\left(n\sigma^2\right)^{-5/2}\cdot 2n\sigma^4\left\{1+(n-1)\bar{\rho}^2\right\} \end{aligned}$$

$$= n^{-1/2}\sigma^{-1} + \frac{3}{4}n^{-3/2}\sigma^{-1}\left\{1+(n-1)\bar{\rho}^2\right\}$$

$$= n^{-1/2}\sigma^{-1}\left\{1+\frac{3}{4}n^{-1}\left\{1+(n-1)\bar{\rho}^2\right\}\right\}$$

$$\frac{1}{\sigma\sqrt{n}}\left\{1+\frac{3}{4}\left\{\frac{1}{n}+\left\{1-\frac{1}{n}\right\}\bar{\rho}^2\right\}\right\}.$$

and so

$$E\left[\frac{\hat{C}p}{Cp}\right] = E\left[\frac{\sigma}{S}\right] = \sigma\sqrt{n}E\left[\frac{1}{\sqrt{Y}}\right] = \sigma\sqrt{n}E\left[g(Y)\right]$$

$$\approx \sigma\sqrt{n}\frac{1}{\sigma\sqrt{n}}\left\{1+\frac{3}{4}\left\{\frac{1}{n}+\left\{1-\frac{1}{n}\right\}\bar{\rho}^2\right\}\right\}$$

$$= 1+\frac{3}{4}\left\{\frac{1}{n}+\left\{1-\frac{1}{n}\right\}\bar{\rho}^2\right\}. \tag{4.60}$$

To get an approximation for $Var\left[\hat{C}p/Cp\right]$, first take $Y = \sum_{i=1}^{n}(X_i-\mu)^2$. We know that $E[Y] = n\sigma^2$ and $Var[Y] = 2n\sigma^4\left\{1+(n-1)\bar{\rho}^2\right\}$. Letting $g(Y) = \frac{1}{\sqrt{Y}} = Y^{-1/2}$ gives $g'(Y) = -\frac{1}{2}Y^{-3/2}$ and $g''(Y) = \frac{3}{4}Y^{-5/2}$. We have

$$Var\left[g(Y)\right] \approx \left\{g'E[Y]\right\}^2 \cdot Var[Y]$$

$$= \left\{-\frac{1}{2}\left(n\sigma^2\right)^{-3/2}\right\}^2 \cdot 2n\sigma^4\left\{1+(n-1)\bar{\rho}^2\right\}$$

$$= \frac{1}{4}\left(n\sigma^2\right)^{-3} 2n\sigma^4\left\{1+(n-1)\bar{\rho}^2\right\}$$

$$= \frac{1}{2n^2\sigma^2}\left\{1+(n-1)\bar{\rho}^2\right\},$$

and so

$$Var\left[\frac{\hat{C}p}{Cp}\right] = Var\left[\frac{\sigma}{S}\right] = n\sigma^2 Var\left[\frac{1}{\sqrt{Y}}\right] = n\sigma^2 Var\left[g(Y)\right]$$

$$\approx n\sigma^2\frac{1}{2n^2\sigma^2}\left\{1+(n-1)\bar{\rho}^2\right\}$$

$$\frac{1}{2n}\left\{1+(n-1)\bar{\rho}^2\right\}$$
$$=\frac{1}{2}\left\{\frac{1}{n}+\left\{1-\frac{1}{n}\right\}\bar{\rho}^2\right\}. \tag{4.61}$$

Collecting results,

$$E\left[\frac{\hat{C}p}{Cp}\right]\approx\hat{E}\left[\frac{\hat{C}p}{Cp}\right]=1+\frac{3}{4}\left\{\frac{1}{n}+\left\{1-\frac{1}{n}\right\}\bar{\rho}^2\right\} \tag{4.62}$$

and

$$Var\left[\frac{\hat{C}p}{Cp}\right]\approx\hat{Var}\left[\frac{\hat{C}p}{Cp}\right]=\frac{1}{2}\left\{\frac{1}{n}+\left\{1-\frac{1}{n}\right\}\bar{\rho}^2\right\}. \tag{4.63}$$

Note that for *iid* $\{X_i\}_n$, $\bar{\rho}=0$ and $\bar{\rho}^2=0$, giving

$$E\left[\frac{\hat{C}p}{Cp}\right]\approx 1+\frac{3}{4n} \quad\text{and}\quad Var\left[\frac{\hat{C}p}{Cp}\right]\approx\frac{1}{2n}, \tag{4.64}$$

our previous results by equations (3.16) and (3.21).

The mean squared error of $\hat{C}p/Cp$ is approximately given by

$$MSE\left[\frac{\hat{C}p}{Cp}\right]=E\left[\left\{\frac{\hat{C}p}{Cp}-1\right\}^2\right]=Var\left[\frac{\hat{C}p}{Cp}\right]+\left\{E\left[\frac{\hat{C}p}{Cp}\right]-1\right\}^2$$
$$\approx\hat{Var}\left[\frac{\hat{C}p}{Cp}\right]+\left\{\hat{E}\left[\frac{\hat{C}p}{Cp}\right]-1\right\}^2$$
$$=\frac{1}{2}\left\{\frac{1}{n}+\left\{1-\frac{1}{n}\right\}\bar{\rho}^2\right\}+\left\{1+\frac{3}{4}\left\{\frac{1}{n}+\left\{1-\frac{1}{n}\right\}\bar{\rho}^2\right\}-1\right\}^2$$
$$=\frac{1}{2}\left\{\frac{1}{n}+\left\{1-\frac{1}{n}\right\}\bar{\rho}^2\right\}+\left\{\frac{3}{4}\left\{\frac{1}{n}+\left\{1-\frac{1}{n}\right\}\bar{\rho}^2\right\}\right\}^2, \tag{4.65}$$

which goes to zero as n grows large. Therefore, the statistic $\hat{C}p$, under autocorrelated observations, is consistent both in mean square and in probability for Cp in the Taylor series approximation.

How accurate are these approximations given by equations (4.62) and (4.63)? We will assess their accuracy for the stationary normal AR(1) model through simulation. We proceed as follows. We fix a sample size n and a parameter ϕ. We generate 10,000 realizations of a time series of length n, $(X_t - \mu) - \phi(X_{t-1} - \mu) = a_t$, where the a_t are *iid* $N(0, \sigma_a^2)$. Without loss of generality, we take μ equal to zero and σ_a^2 equal to one. We take X_0 equal to zero. For each of the 10,000 generated series, we calculate

$$\frac{1}{S} = \left\{ \frac{1}{n} \sum_{i=1}^{n} (X_i - 0)^2 \right\}^{-1/2}, \tag{4.66}$$

resulting in 10,000 realizations of $1/S$. We next find the mean and variance of the 10,000 values of $1/S$ by

$$\tilde{E}\left[\frac{1}{S}\right] = \frac{1}{10000} \sum_{j=1}^{10000} \left(\frac{1}{S_j}\right) \tag{4.67}$$

and

$$\tilde{Var}\left[\frac{1}{S}\right] = \frac{1}{10000} \sum_{j=1}^{10000} \left(\frac{1}{S_j}\right)^2 - \left(\frac{1}{10000} \sum_{j=1}^{10000} \left(\frac{1}{S_j}\right)\right)^2. \tag{4.68}$$

We then determine, for $\hat{\theta} = \hat{C}p/Cp$,

$$\tilde{E}[\hat{\theta}] = \tilde{E}\left[\frac{\hat{C}p}{Cp}\right] = \tilde{E}\left[\frac{\sigma}{S}\right] = \sigma \tilde{E}\left[\frac{1}{S}\right] = \sqrt{\frac{1}{1-\phi^2}}\, \tilde{E}\left[\frac{1}{S}\right] \tag{4.69}$$

and

$$\tilde{Var}[\hat{\theta}] = \tilde{Var}\left[\frac{\hat{C}p}{Cp}\right] = \tilde{Var}\left[\frac{\sigma}{S}\right] = \sigma^2 \tilde{Var}\left[\frac{1}{S}\right] = \frac{1}{1-\phi^2} \tilde{Var}\left[\frac{1}{S}\right]. \tag{4.70}$$

The following computational device will ease the calculation of our Taylor approximated mean and variance for the AR(1) model. Suppose a stochastic process follows an AR(1), that is, $(X_t - \mu) - \phi(X_{t-1} - \mu) = a_t$, where the a_t are *iid* $N(0, \sigma_a^2)$. The correlation j

periods apart is given by $Corr\left[X_t, X_{t-j}\right] = \phi^j$. Consider a sample $\{X_t\}_n$ from an AR(1) process, consecutive in time, taken at the uniform time interval consistent with the parameter ϕ. Let $S^2 = \frac{1}{n}\sum_{t=1}^{n}(X_t - \mu)^2$ be the sample variance. In Appendix C, we show that

$$\bar{\rho} = \frac{2}{n(n-1)} \sum_{i<j}^{n(n-1)/2} \rho_{ij} = \frac{2}{n(n-1)} \frac{\phi}{1-\phi}\left\{n - \frac{1-\phi^n}{1-\phi}\right\} \tag{4.71}$$

and

$$\bar{\rho}^2 = \frac{2}{n(n-1)} \sum_{i<j}^{n(n-1)/2} \rho_{ij}^2 = \frac{2}{n(n-1)} \frac{\phi^2}{1-\phi^2}\left\{n - \frac{1-\phi^{2n}}{1-\phi^2}\right\}. \tag{4.72}$$

Therefore, for a sample of size n from the AR(1), we have

$$E\left[\frac{\hat{C}p}{Cp}\right] \approx \hat{E}\left[\frac{\hat{C}p}{Cp}\right] = 1 + \frac{3}{4}\left\{\frac{1}{n} + \left\{1 - \frac{1}{n}\right\}\left\{\frac{2}{n(n-1)} \frac{\phi^2}{1-\phi^2}\left\{n - \frac{1-\phi^{2n}}{1-\phi^2}\right\}\right\}\right\} \quad \text{and}$$

$$Var\left[\frac{\hat{C}p}{Cp}\right] \approx \hat{Var}\left[\frac{\hat{C}p}{Cp}\right] = \frac{1}{2}\left\{\frac{1}{n} + \left\{1 - \frac{1}{n}\right\}\left\{\frac{2}{n(n-1)} \frac{\phi^2}{1-\phi^2}\left\{n - \frac{1-\phi^{2n}}{1-\phi^2}\right\}\right\}\right\}. \tag{4.73}$$

Note that for ϕ equal to zero, we get our previous results of equations (3.16) and (3.21) for $iid\ \{X_t\}_n$.

Table 4.8 compares the *simulated* mean $\tilde{E}\left[\hat{\theta}\right]$ and variance $\tilde{Var}\left[\hat{\theta}\right]$ of $\hat{\theta} = \hat{C}p/Cp$ with the *Taylor-series-approximated* mean $\hat{E}\left[\hat{\theta}\right]$ and variance $\hat{Var}\left[\hat{\theta}\right]$ of $\hat{\theta} = \hat{C}p/Cp$ for sample sizes n and ϕ of the AR(1) model. A tilde over the expectation or variance operator indicates the simulated value, while a carat over the operator indicates the Taylor approximated value. We denote the relative errors by

$$\left(\tilde{E} - \hat{E}\right)/\tilde{E} = \frac{\tilde{E}\left[\hat{\theta}\right] - \hat{E}\left[\hat{\theta}\right]}{\tilde{E}\left[\hat{\theta}\right]} = 1 - \frac{\hat{E}\left[\hat{\theta}\right]}{\tilde{E}\left[\hat{\theta}\right]},$$

Table 4.8. Simulate vs. Approximate Mean, Variance of $\hat{\theta} = \hat{Cp}/Cp$, Known μ

n	ϕ	$\tilde{E}[\hat{\theta}]$	$\hat{E}[\hat{\theta}]$	$(\tilde{E}-\hat{E})/\tilde{E}$	$\tilde{Var}[\hat{\theta}]$	$\hat{Var}[\hat{\theta}]$	$(\tilde{V}-\hat{V})/\tilde{V}$
10	0.1	1.08899	1.07636	0.011598	0.077640	0.050908	0.344307
10	0.2	1.09435	1.08060	0.012564	0.081370	0.053733	0.339646
10	0.4	1.11988	1.10017	0.017600	0.099141	0.066780	0.326414
10	0.5	1.14415	1.11833	0.022567	0.116428	0.078889	0.322422
10	0.6	1.18235	1.14619	0.030583	0.144469	0.097461	0.325385
10	0.8	1.37039	1.26845	0.074388	0.295602	0.178964	0.394578
10	0.9	1.69762	1.41883	0.164224	0.605689	0.279218	0.539008
20	0.1	1.04556	1.03822	0.007020	0.031752	0.025480	0.197531
20	0.2	1.04854	1.04046	0.007706	0.033502	0.026975	0.194824
20	0.4	1.06263	1.05094	0.011001	0.041691	0.033957	0.185508
20	0.5	1.07602	1.06083	0.014117	0.049653	0.040556	0.183212
20	0.6	1.09721	1.07639	0.018975	0.062574	0.050928	0.186116
20	0.8	1.20398	1.15232	0.042908	0.133775	0.101545	0.240927
20	0.9	1.40005	1.27434	0.089790	0.284837	0.182893	0.357903
30	0.1	1.02945	1.02549	0.003847	0.019540	0.016992	0.130399
30	0.2	1.03152	1.02701	0.004372	0.020645	0.018007	0.127779
30	0.4	1.04120	1.03415	0.006771	0.025887	0.022764	0.120640
30	0.5	1.05043	1.04093	0.009044	0.030995	0.027284	0.119729
30	0.6	1.06508	1.05166	0.012600	0.039372	0.034440	0.125267
30	0.8	1.13976	1.10566	0.029919	0.086144	0.070439	0.182311
30	0.9	1.28027	1.20083	0.062049	0.187323	0.133886	0.285267
40	0.1	1.02192	1.01912	0.002740	0.013946	0.012746	0.086046
40	0.2	1.02352	1.02027	0.003175	0.014714	0.013515	0.081487
40	0.4	1.03105	1.02568	0.005208	0.018458	0.017120	0.072489
40	0.5	1.03821	1.03083	0.007108	0.022135	0.020556	0.071335
40	0.6	1.04955	1.03902	0.010033	0.028133	0.026013	0.075356
40	0.8	1.10717	1.08079	0.023826	0.061374	0.053858	0.122462
40	0.9	1.21614	1.15759	0.048144	0.132892	0.105058	0.209448
50	0.1	1.01732	1.01530	0.001986	0.011019	0.010198	0.074508
50	0.2	1.01863	1.01622	0.002366	0.011629	0.010816	0.069911
50	0.4	1.02473	1.02058	0.004050	0.014626	0.013719	0.062013
50	0.5	1.03055	1.02473	0.005648	0.017599	0.016489	0.063072
50	0.6	1.03980	1.03135	0.008127	0.022462	0.020898	0.069629
50	0.8	1.08708	1.06537	0.019971	0.049462	0.043580	0.118920
50	0.9	1.17781	1.12943	0.041076	0.107648	0.086288	0.198425

$$\left(\tilde{V}-\hat{V}\right)/\tilde{V}=\frac{\tilde{Var}\left[\hat{\theta}\right]-\hat{Var}\left[\hat{\theta}\right]}{\tilde{Var}\left[\hat{\theta}\right]}=1-\frac{\hat{Var}\left[\hat{\theta}\right]}{\tilde{Var}\left[\hat{\theta}\right]}.$$

We first address the two columns of Table 4.8 labelled $\tilde{E}\left[\hat{\theta}\right]$ and $\tilde{Var}\left[\hat{\theta}\right]$. If we accept the simulated $\tilde{E}\left[\hat{\theta}\right]$ and simulated $\tilde{Var}\left[\hat{\theta}\right]$ as close to the true mean and variance of $\hat{\theta}=\hat{C}p/Cp$, we see that each decreases as n increases, but each increases as ϕ increases. This is to be expected. The simulated $\tilde{E}\left[\hat{\theta}\right]$ range from a low of 1.01732 at the point $(n, \phi) = (50, 0.1)$ to a high of 1.69762 at $(n, \phi) = (10, 0.9)$. Recall that 1.69762 indicates that $\hat{C}p$ is overestimating the true Cp by almost 70 percent on average. The simulated $\tilde{Var}\left[\hat{\theta}\right]$ range from a low of 0.011019 at $(n, \phi) = (50, 0.1)$ to a high of 0.605689 at $(n, \phi) = (10, 0.9)$.

The Taylor-series-approximated mean $\hat{E}\left[\hat{\theta}\right]$ and variance $\hat{Var}\left[\hat{\theta}\right]$ follow the same pattern, that is, each decreases as n increases, but each increases as ϕ increases. This is a good sign, of course, indicating that these approximations are generally tracking the true parameters.

We now evaluate our Taylor-series-approximated mean $\hat{E}\left[\hat{\theta}\right]$ and variance $\hat{Var}\left[\hat{\theta}\right]$ of $\hat{\theta}=\hat{C}p/Cp$ through their errors relative to the simulated values. We look to the columns of Table 4.8 labelled $\left(\tilde{E}-\hat{E}\right)/\tilde{E}$ and $\left(\tilde{V}-\hat{V}\right)/\tilde{V}$. Our mean $\hat{E}\left[\hat{\theta}\right]$ performed well except at the point $(n, \phi) = (10, 0.9)$, where it has a relative error of 0.164224. At all other points (n, ϕ), our $\hat{E}\left[\hat{\theta}\right]$ has a relative error of less than 0.10 and most are much smaller than 0.10. Note that $\left(\tilde{E}-\hat{E}\right)/\tilde{E}$ decreases as n increases, but increases as ϕ increases. Also note that all $\left(\tilde{E}-\hat{E}\right)/\tilde{E}$ are positive, indicating that the Taylor-series-approximated mean $\hat{E}\left[\hat{\theta}\right]$ is consistently underestimating the true mean.

Our Taylor-series-approximated variance $\hat{Var}\left[\hat{\theta}\right]$ did not perform as well. It did not achieve a relative error below 0.10 until $n = 40$. Its relative error also displays a curious bathtub shape in ϕ for all sample sizes n, where it reaches a relative minimum at $\phi = 0.5$. Note also that all $\left(\tilde{V} - \hat{V}\right)/\tilde{V}$ are positive, indicating that the Taylor-series-approximated variance $\hat{Var}\left[\hat{\theta}\right]$ is consistently underestimating the true variance. This is to be expected since the finite Taylor series expansion truncates positive terms.

4.6. Estimating *Cp* when Both μ and σ are Unknown

Let $\{X_i\}_n$ be multivariate normal with equal means μ, equal variances σ^2, and correlations $\rho(i, j)$ not necessarily zero. In vector notation, let $\mathbf{x}$ be a random n-vector with mean vector μ with identical coordinates, correlation matrix $\mathbf{R}$, and covariance matrix $\sigma^2\mathbf{R}$. In this section we assume that the common process mean μ is unknown and so we use $S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2}$ in the definition of $\hat{C}p$. Recall that we have a lower bound for $E\left[\hat{C}p/Cp\right]$ given by

$$\frac{1}{\sqrt{1-\overline{\rho}}} < E\left[\frac{\hat{C}p}{Cp}\right],$$

where

$$\overline{\rho} = \frac{1}{n(n-1)}\sum_{i \neq j}^{n(n-1)} \rho_{ij} = \frac{2}{n(n-1)}\sum_{i<j}^{n(n-1)/2} \rho_{ij}$$

is the average of the $n(n - 1)$ pairwise correlation parameters of the model.

We will derive the Taylor series approximation for $E\left[\hat{C}p/Cp\right]$,

$$E[g(Y)] \approx gE[Y] + \frac{1}{2}g''E[Y] \cdot Var[Y],$$

where we first let $Y = \sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2$. We will need both $E[Y] = E\left[\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2\right]$ and $Var[Y] = Var\left[\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2\right]$. From previous results,

$$E\left[\sum_{i=1}^{n}(X_i-\bar{X})^2\right]=(n-1)(1-\bar{\rho})\sigma^2=\sigma^2 trace\mathbf{CR}, \tag{4.74}$$

where $\mathbf{C}=\mathbf{I}-\frac{1}{n}\mathbf{11}'$ is the centering matrix and

$$\bar{\rho}=\frac{1}{n(n-1)}\sum_{i\neq j}^{n(n-1)}\rho_{ij}=\frac{2}{n(n-1)}\sum_{i<j}^{n(n-1)/2}\rho_{ij}$$

is the average of the $n(n-1)$ pairwise correlation parameters of the model.

At equation (D.23) of Appendix D, we derive an expression for $Var\left[\sum_{i=1}^{n}(X_i-\bar{X})^2\right]$ given by

$$2\sigma^4\left\{\begin{array}{l}(n-1)-\frac{2}{n}\sum_{i\neq j}^{n(n-1)}\rho_{ij}+\left\{1-\frac{2}{n}\right\}\sum_{i\neq j}^{n(n-1)}\rho_{ij}^2\\ -\frac{2}{n}\sum_{i\neq j\neq k}^{n(n-1)(n-2)}\rho_{ij}\rho_{ik}+\left\{\frac{1}{n}\right\}^2\sum_{i\neq j}^{n(n-1)}\rho_{ij}\sum_{i\neq j}^{n(n-1)}\rho_{ij}\end{array}\right\}, \tag{4.75}$$

which we denote by $2\sigma^4 trace\mathbf{CRCR}$. Note that $trace\mathbf{CRCR}$ is the sum of the terms within the braces. The justification for this notation is at equation (D.23) of Appendix D.

Recall the Taylor series approximation,

$$E[g(Y)]\approx gE[Y]+\frac{1}{2}g''E[Y]\cdot Var[Y]. \tag{4.76}$$

To get $E\left[\hat{C}p/Cp\right]$, first take $Y=\sum_{i=1}^{n}(X_i-\bar{X})^2$. We know that

$$E[Y]=E\left[\sum_{i=1}^{n}(X_i-\bar{X})^2\right]=(n-1)(1-\bar{\rho})\sigma^2=\sigma^2 trace\mathbf{CR} \tag{4.77}$$

and

$$Var[Y]=Var\left[\sum_{i=1}^{n}(X_i-\bar{X})^2\right]=2\sigma^4 trace\mathbf{CRCR}. \tag{4.78}$$

Letting $g(Y)=\frac{1}{\sqrt{Y}}=Y^{-1/2}$ gives $g'(Y)=-\frac{1}{2}Y^{-3/2}$ and $g''(Y)=\frac{3}{4}Y^{-5/2}$. We have

$$E[g(Y)]\approx gE[Y]+\frac{1}{2}g''E[Y]\cdot Var[Y]$$

$$=\left(\sigma^2 trace\mathbf{CR}\right)^{-1/2}+\frac{1}{2}\frac{3}{4}\left(\sigma^2 trace\mathbf{CR}\right)^{-5/2}2\sigma^4 trace\mathbf{CRCR}$$

$$=\frac{1}{\sigma}\left\{(trace\mathbf{CR})^{-1/2}+\frac{3}{4}(trace\mathbf{CR})^{-5/2}trace\mathbf{CRCR}\right\}$$

$$=\frac{1}{\sigma}(trace\mathbf{CR})^{-1/2}\left\{1+\frac{3}{4}(trace\mathbf{CR})^{-2}trace\mathbf{CRCR}\right\},$$

and so

$$E\left[\frac{\hat{C}p}{Cp}\right]=E\left[\frac{\sigma}{S}\right]=\sigma\sqrt{n-1}E\left[\frac{1}{\sqrt{Y}}\right]=\sigma\sqrt{n-1}E[g(Y)]$$

$$\approx\sigma\sqrt{n-1}\frac{1}{\sigma}(trace\mathbf{CR})^{-1/2}\left\{1+\frac{3}{4}(trace\mathbf{CR})^{-2}trace\mathbf{CRCR}\right\}$$

$$=\sqrt{n-1}(trace\mathbf{CR})^{-1/2}\left\{1+\frac{3}{4}(trace\mathbf{CR})^{-2}trace\mathbf{CRCR}\right\}$$

$$=\sqrt{n-1}\{(n-1)(1-\bar{\rho})\}^{-1/2}\left\{1+\frac{3}{4}\{(n-1)(1-\bar{\rho})\}^{-2}trace\mathbf{CRCR}\right\}$$

$$=\frac{1}{\sqrt{1-\bar{\rho}}}\left\{1+\frac{3}{4}\{(n-1)(1-\bar{\rho})\}^{-2}trace\mathbf{CRCR}\right\}$$

$$=\frac{1}{\sqrt{1-\bar{\rho}}}\left\{1+\frac{3}{4}\frac{trace\mathbf{CRCR}}{trace\mathbf{CR}\;trace\mathbf{CR}}\right\}.$$

Therefore,

$$E\left[\frac{\hat{C}p}{Cp}\right]\approx\hat{E}\left[\frac{\hat{C}p}{Cp}\right]=\frac{1}{\sqrt{1-\bar{\rho}}}\left\{1+\frac{3}{4}\frac{trace\mathbf{CRCR}}{trace\mathbf{CR}\;trace\mathbf{CR}}\right\}. \tag{4.79}$$

Interestingly, the lower bound $\frac{1}{\sqrt{1-\bar{\rho}}}<E\left[\frac{\hat{C}p}{Cp}\right]$, which we have previously derived, appears as the first factor in this Taylor series approximation. We also point out the resemblance of equation (4.79) to each of equations (4.62) and (3.36).

To get $Var[\hat{C}p/Cp]$, first take $Y = \sum_{i=1}^{n}(X_i - \mu)^2$. Letting $g(Y) = \frac{1}{\sqrt{Y}} = Y^{-1/2}$ gives $g'(Y) = -\frac{1}{2}Y^{-3/2}$ and $g''(Y) = \frac{3}{4}Y^{-5/2}$. We have

$$Var[g(Y)] \approx \{g'E[Y]\}^2 \cdot Var[Y] \tag{4.80}$$

$$= \left\{-\frac{1}{2}\left(\sigma^2 trace\mathbf{CR}\right)^{-3/2}\right\}^2 2\sigma^4 trace\mathbf{CRCR}$$

$$= \frac{1}{2\sigma^2}(trace\mathbf{CR})^{-3} trace\mathbf{CRCR},$$

and so

$$Var\left[\frac{\hat{C}p}{Cp}\right] = Var\left[\frac{\sigma}{S}\right] = (n-1)\sigma^2\, Var\left[\frac{1}{\sqrt{Y}}\right] = (n-1)\sigma^2\, Var[g(Y)]$$

$$\approx (n-1)\sigma^2 \frac{1}{2\sigma^2}(trace\mathbf{CR})^{-3} trace\mathbf{CRCR}$$

$$= (n-1)\sigma^2 \frac{1}{2\sigma^2}\{(n-1)(1-\bar{\rho})\}^{-3} trace\mathbf{CRCR}$$

$$= \frac{trace\mathbf{CRCR}}{2(n-1)^2(1-\bar{\rho})^3}$$

$$= \frac{1}{2(1-\bar{\rho})}\frac{trace\mathbf{CRCR}}{trace\mathbf{CR}\; trace\mathbf{CR}}.$$

Therefore,

$$Var\left[\frac{\hat{C}p}{Cp}\right] \approx \hat{Var}\left[\frac{\hat{C}p}{Cp}\right] = \frac{1}{2(1-\bar{\rho})}\frac{trace\mathbf{CRCR}}{trace\mathbf{CR}\; trace\mathbf{CR}}. \tag{4.81}$$

Collecting results, we have

$$E\left[\frac{\hat{C}p}{Cp}\right] \approx \hat{E}\left[\frac{\hat{C}p}{Cp}\right] = \frac{1}{\sqrt{1-\bar{\rho}}}\left\{1 + \frac{3}{4}\frac{trace\mathbf{CRCR}}{trace\mathbf{CR}\; trace\mathbf{CR}}\right\} \tag{4.82}$$

and

$$Var\left[\frac{\hat{C}p}{Cp}\right] \approx \hat{Var}\left[\frac{\hat{C}p}{Cp}\right] = \frac{1}{2(1-\bar{\rho})}\frac{trace\mathbf{CRCR}}{trace\mathbf{CR}\; trace\mathbf{CR}}. \tag{4.83}$$

Note that in the case of uncorrelated characteristics $\{X_i\}_n$,

$$E\left[\frac{\hat{C}p}{Cp}\right] \approx \frac{1}{\sqrt{1-\bar{\rho}}}\left\{1+\frac{3}{4}\frac{trace\mathbf{CRCR}}{trace\mathbf{CR}\; trace\mathbf{CR}}\right\}$$

$$= \frac{1}{\sqrt{1-0}}\left\{1+\frac{3}{4}\frac{trace\mathbf{CICI}}{trace\mathbf{CI}\; trace\mathbf{CI}}\right\} = 1+\frac{3}{4}\frac{trace\mathbf{CC}}{trace\mathbf{C}\; trace\mathbf{C}}$$

$$= 1+\frac{3}{4}\frac{trace\mathbf{C}}{trace\mathbf{C}\; trace\mathbf{C}} = 1+\frac{3}{4trace\mathbf{C}} = 1+\frac{3}{4(n-1)} \tag{4.84}$$

and

$$Var\left[\frac{\hat{C}p}{Cp}\right] \approx \frac{1}{2(1-\bar{\rho})}\frac{trace\mathbf{CRCR}}{trace\mathbf{CR}\; trace\mathbf{CR}} = \frac{1}{2(1-0)}\frac{trace\mathbf{CICI}}{trace\mathbf{CI}\; trace\mathbf{CI}}$$

$$= \frac{1}{2}\frac{trace\mathbf{CC}}{trace\mathbf{C}\; trace\mathbf{C}} = \frac{1}{2}\frac{trace\mathbf{C}}{trace\mathbf{C}\; trace\mathbf{C}}$$

$$= \frac{1}{2trace\mathbf{C}} = \frac{1}{2(n-1)}, \tag{4.85}$$

since $\mathbf{C} = \mathbf{I} - \frac{1}{n}\mathbf{1}\mathbf{1}'$ is idempotent with its rank equal to its trace, each equaling $(n - 1)$. This is consistent with our previous equations (3.36) and (3.37) for the case of independent, identically normal characteristics $\{X_i\}_n$.

How accurate are these approximations given by equations (4.82) and (4.83)? We will assess their accuracy for the stationary normal AR(1) model through simulation. We proceed as follows. We fix a sample size n and a parameter ϕ. We generate 10,000 realizations of a time series of length n, $(X_t-\mu)-\phi(X_{t-1}-\mu)=a_t$, where the a_t are *iid* $N\left(0,\sigma_a^2\right)$. Without loss of generality, we take μ equal to zero and σ_a^2 equal to one. We take X_0 equal to zero. For each of the 10,000 generated series, we calculate

$$\frac{1}{S} = \left\{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2\right\}^{-1/2}, \tag{4.86}$$

resulting in 10,000 realizations of $1/S$. We next find the mean and variance of the 10,000 values of $1/S$ by

$$\tilde{E}\left[\frac{1}{S}\right]=\frac{1}{10000}\sum_{j=1}^{10000}\left(\frac{1}{S_j}\right) \tag{4.87}$$

and

$$\tilde{V}ar\left[\frac{1}{S}\right]=\frac{1}{10000}\sum_{j=1}^{10000}\left(\frac{1}{S_j}\right)^2-\left(\frac{1}{10000}\sum_{j=1}^{10000}\left(\frac{1}{S_j}\right)\right)^2. \tag{4.88}$$

We then determine, for $\hat{\theta}=\hat{C}p/Cp$,

$$\tilde{E}\left[\hat{\theta}\right]=\tilde{E}\left[\frac{\hat{C}p}{Cp}\right]=\tilde{E}\left[\frac{\sigma}{S}\right]=\sigma\tilde{E}\left[\frac{1}{S}\right]=\sqrt{\frac{1}{1-\phi^2}}\tilde{E}\left[\frac{1}{S}\right] \tag{4.89}$$

and

$$\tilde{V}ar\left[\hat{\theta}\right]=\tilde{V}ar\left[\frac{\hat{C}p}{Cp}\right]=\tilde{V}ar\left[\frac{\sigma}{S}\right]=\sigma^2\tilde{V}ar\left[\frac{1}{S}\right]=\frac{1}{1-\phi^2}\tilde{V}ar\left[\frac{1}{S}\right]. \tag{4.90}$$

Table 4.9 compares the *simulated* mean $\tilde{E}\left[\hat{\theta}\right]$ and variance $\tilde{V}ar\left[\hat{\theta}\right]$ of $\hat{\theta}=\hat{C}p/Cp$ with the *Taylor-series-approximated* mean $\hat{E}\left[\hat{\theta}\right]$ and variance $\hat{V}ar\left[\hat{\theta}\right]$ of $\hat{\theta}=\hat{C}p/Cp$ for sample sizes n and ϕ of the AR(1) model. A tilde over the expectation or variance operator indicates the simulated value, while a carat over the operator indicates the Taylor approximated value. We denote the relative errors by

$$\left(\tilde{E}-\hat{E}\right)/\tilde{E}=\frac{\tilde{E}\left[\hat{\theta}\right]-\hat{E}\left[\hat{\theta}\right]}{\tilde{E}\left[\hat{\theta}\right]}=1-\frac{\hat{E}\left[\hat{\theta}\right]}{\tilde{E}\left[\hat{\theta}\right]},$$

$$\left(\tilde{V}-\hat{V}\right)/\tilde{V}=\frac{\tilde{V}ar\left[\hat{\theta}\right]-\hat{V}ar\left[\hat{\theta}\right]}{\tilde{V}ar\left[\hat{\theta}\right]}=1-\frac{\hat{V}ar\left[\hat{\theta}\right]}{\tilde{V}ar\left[\hat{\theta}\right]}.$$

We first address the two columns of Table 4.9 labelled $\tilde{E}\left[\hat{\theta}\right]$ and $\tilde{V}ar\left[\hat{\theta}\right]$. If we accept the simulated $\tilde{E}\left[\hat{\theta}\right]$ and simulated $\tilde{V}ar\left[\hat{\theta}\right]$ as close to the true mean and variance of $\hat{\theta}=\hat{C}p/Cp$,

Table 4.9. Simulate vs. Approximate Mean, Variance of $\hat{\theta} = \hat{Cp}/Cp$, Unknown μ

n	ϕ	$\tilde{E}[\hat{\theta}]$	$\hat{E}[\hat{\theta}]$	$(\tilde{E}-\hat{E})/\tilde{E}$	$\tilde{Var}[\hat{\theta}]$	$\hat{Var}[\hat{\theta}]$	$(\tilde{V}-\hat{V})/\tilde{V}$
10	0.1	1.11189	1.09674	0.013625	0.093547	0.057690	0.383305
10	0.2	1.13249	1.11607	0.014499	0.099676	0.062090	0.377082
10	0.4	1.20359	1.18120	0.018603	0.125207	0.080528	0.356841
10	0.5	1.26364	1.23570	0.022111	0.148962	0.097672	0.344316
10	0.6	1.35334	1.31720	0.026704	0.186446	0.124839	0.330428
10	0.8	1.74926	1.68780	0.035135	0.377850	0.266189	0.295517
10	0.9	2.33702	2.27278	0.027488	0.775707	0.551420	0.289139
20	0.1	1.05370	1.04598	0.007327	0.033814	0.027090	0.198853
20	0.2	1.06407	1.05550	0.008054	0.036026	0.028978	0.195637
20	0.4	1.10064	1.08813	0.011366	0.045880	0.037445	0.183849
20	0.5	1.13228	1.11609	0.014299	0.055385	0.045587	0.176907
20	0.6	1.18095	1.15906	0.018536	0.070857	0.058805	0.170089
20	0.8	1.41767	1.37125	0.032744	0.153647	0.130063	0.153495
20	0.9	1.81075	1.74331	0.037244	0.318028	0.275261	0.134476
30	0.1	1.03431	1.03016	0.004012	0.020332	0.017696	0.129648
30	0.2	1.04123	1.03646	0.004581	0.021605	0.018886	0.125850
30	0.4	1.06567	1.05813	0.007075	0.027404	0.024324	0.112392
30	0.5	1.08687	1.07674	0.009320	0.033045	0.029580	0.104857
30	0.6	1.11967	1.10552	0.012638	0.042289	0.038147	0.097945
30	0.8	1.28483	1.25183	0.025684	0.093908	0.085291	0.091760
30	0.9	1.58237	1.52484	0.036357	0.202418	0.183239	0.094749
40	0.1	1.02511	1.02244	0.002605	0.014500	0.013139	0.093862
40	0.2	1.03026	1.02715	0.003019	0.015411	0.014006	0.091169
40	0.4	1.04858	1.04335	0.004988	0.019647	0.018000	0.083830
40	0.5	1.06453	1.05727	0.006820	0.023852	0.021864	0.083347
40	0.6	1.08928	1.07882	0.009603	0.030812	0.028157	0.086168
40	0.8	1.21559	1.18960	0.021381	0.070183	0.063018	0.102090
40	0.9	1.45288	1.40397	0.033664	0.152419	0.136758	0.102750
50	0.1	1.01961	1.01786	0.001716	0.011228	0.010448	0.069469
50	0.2	1.02371	1.02162	0.002042	0.011885	0.011129	0.063610
50	0.4	1.03823	1.03456	0.003535	0.015070	0.014283	0.052223
50	0.5	1.05089	1.04567	0.004967	0.018244	0.017332	0.049989
50	0.6	1.07056	1.06287	0.007183	0.023527	0.022293	0.052450
50	0.8	1.17164	1.15168	0.017036	0.053812	0.049785	0.074835
50	0.9	1.36750	1.32727	0.029419	0.119234	0.108648	0.088783

we see that each decreases as n increases, but each increases as ϕ increases. This is to be expected. The simulated $\tilde{E}\left[\hat{\theta}\right]$ range from a low of 1.01961 at the point $(n, \phi) = (50, 0.1)$ to a high of 2.33702 at $(n, \phi) = (10, 0.9)$. The simulated $\tilde{Var}\left[\hat{\theta}\right]$ range from a low of 0.011228 at $(n, \phi) = (50, 0.1)$ to a high of 0.775707 at $(n, \phi) = (10, 0.9)$.

The Taylor-series-approximated mean $\hat{E}\left[\hat{\theta}\right]$ and variance $\hat{Var}\left[\hat{\theta}\right]$ follow the same pattern, that is, each decreases as n increases, but each increases as ϕ increases. Again, this indicates that these approximations are generally tracking the true parameters.

We now evaluate our Taylor-series-approximated mean $\hat{E}\left[\hat{\theta}\right]$ and variance $\hat{Var}\left[\hat{\theta}\right]$ of $\hat{\theta} = \hat{Cp}/Cp$ through their errors relative to the simulated values. We look to the columns of Table 4.9 labelled $\left(\tilde{E} - \hat{E}\right)/\tilde{E}$ and $\left(\tilde{V} - \hat{V}\right)/\tilde{V}$. Our mean $\hat{E}\left[\hat{\theta}\right]$ performed well for all (n, ϕ), with a relative error of less than 0.05. Note that $\left(\tilde{E} - \hat{E}\right)/\tilde{E}$ decreases as n increases, but increases as ϕ increases. Also note that all $\left(\tilde{E} - \hat{E}\right)/\tilde{E}$ are positive, indicating that the Taylor-series-approximated mean $\hat{E}\left[\hat{\theta}\right]$ is consistently underestimating the true mean.

Our Taylor-series-approximated variance $\hat{Var}\left[\hat{\theta}\right]$ did not perform as well. It did not achieve a relative error below 0.10 until $n = 30$. Its relative error again displays a curious bathtub shape in ϕ for $n = 40$ and 50, where it reaches a relative minimum at $\phi = 0.5$. Note also that all $\left(\tilde{V} - \hat{V}\right)/\tilde{V}$ are positive, indicating that the Taylor-series-approximated variance $\hat{Var}\left[\hat{\theta}\right]$ is consistently underestimating the true variance.

4.7. Estimating (*Cpl*, *Cp*, *Cpu*) when μ is Unknown and σ is Known

Consider the natural estimators

$$\left(\hat{Cpl}, \hat{Cp}, \hat{Cpu}\right) = \left(\frac{\overline{X} - LSL}{3\sigma}, \frac{USL - LSL}{6\sigma}, \frac{USL - \overline{X}}{3\sigma}\right), \tag{4.91}$$

where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Note that here, $\hat{C}p = Cp$ is a known constant.

We will need $Var[\bar{X}] = Var\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right]$. First,

$$Var\left[\sum_{i=1}^{n} X_i\right] = E\left[\sum_{i=1}^{n} X_i \sum_{i=1}^{n} X_i\right] - E^2\left[\sum_{i=1}^{n} X_i\right]$$

$$= E\left[\sum_{i=1}^{n} X_i^2 + \sum_{i \neq j}^{n(n-1)} X_i X_j\right] - E^2\left[\sum_{i=1}^{n} X_i\right]$$

$$= E\left[\sum_{i=1}^{n} X_i^2\right] + E\left[\sum_{i \neq j}^{n(n-1)} X_i X_j\right] - E^2\left[\sum_{i=1}^{n} X_i\right]$$

$$= n\left(\sigma^2 + \mu^2\right) + \sigma^2 \sum_{i \neq j}^{n(n-1)} \rho_{ij} + n(n-1)\mu^2 - n^2\mu^2$$

$$= n\sigma^2 + n(n-1)\sigma^2\bar{\rho}$$

$$= n\sigma^2\{1 + (n-1)\bar{\rho}\}, \tag{4.92}$$

and so

$$Var[\bar{X}] = Var\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n^2} Var\left[\sum_{i=1}^{n} X_i\right]$$

$$= \frac{1}{n^2} n\sigma^2\{1 + (n-1)\bar{\rho}\} = \frac{\sigma^2}{n}\{1 + (n-1)\bar{\rho}\}$$

$$= \sigma^2\left\{\frac{1}{n} + \left\{1 - \frac{1}{n}\right\}\bar{\rho}\right\}. \tag{4.93}$$

Now for unknown μ and known σ, we have

$$E\left[\hat{C}pl\right] = E\left[\frac{\bar{X} - LSL}{3\sigma}\right] = \frac{\mu - LSL}{3\sigma} = Cpl \tag{4.94}$$

and

$$Var\left[\hat{C}pl\right] = Var\left[\frac{\bar{X} - LSL}{3\sigma}\right] = \frac{1}{9\sigma^2} Var\left[\bar{X} - LSL\right] = \frac{1}{9\sigma^2} Var\left[\bar{X}\right]$$

$$= \frac{1}{9\sigma^2}\sigma^2\left\{\frac{1}{n}+\left\{1-\frac{1}{n}\right\}\bar{\rho}\right\} = \frac{1}{9}\left\{\frac{1}{n}+\left\{1-\frac{1}{n}\right\}\bar{\rho}\right\}. \tag{4.95}$$

Similarly,

$$E\left[\hat{C}pu\right] = Cpu \quad \text{and} \quad Var\left[\hat{C}pu\right] = \frac{1}{9}\left\{\frac{1}{n}+\left\{1-\frac{1}{n}\right\}\bar{\rho}\right\}. \tag{4.96}$$

The covariance can be gotten as

$$Cov\left[\hat{C}pl, \hat{C}pu\right] = Cov\left[\frac{\bar{X}-LSL}{3\sigma}, \frac{USL-\bar{X}}{3\sigma}\right] = \frac{1}{9\sigma^2}Cov\left[\bar{X}-LSL, USL-\bar{X}\right]$$

$$= \frac{1}{9\sigma^2}Cov\left[\bar{X}, -\bar{X}\right] = -\frac{1}{9\sigma^2}Cov\left[\bar{X}, \bar{X}\right] = -\frac{1}{9\sigma^2}Var\left[\bar{X}\right].$$

$$= -\frac{1}{9\sigma^2}\sigma^2\left\{\frac{1}{n}+\left\{1-\frac{1}{n}\right\}\bar{\rho}\right\} = -\frac{1}{9}\left\{\frac{1}{n}+\left\{1-\frac{1}{n}\right\}\bar{\rho}\right\}. \tag{4.97}$$

Of course, we have

$$Corr\left[\hat{C}pl, \hat{C}pu\right] = \frac{Cov\left[\hat{C}pl, \hat{C}pu\right]}{\sqrt{Var\left[\hat{C}pl\right]}\sqrt{Var\left[\hat{C}pu\right]}} = -1. \tag{4.98}$$

Now since $\hat{C}p = Cp$ is a known constant, we should observe

$$E\left[\hat{C}p\right] = E[Cp] = Cp \quad \text{and} \quad Var\left[\hat{C}p\right] = Var[Cp] = 0, \tag{4.99}$$

which we confirm by

$$E\left[\hat{C}p\right] = E\left[\frac{1}{2}\left(\hat{C}pl+\hat{C}pu\right)\right] = \frac{1}{2}E\left[\hat{C}pl\right]+\frac{1}{2}E\left[\hat{C}pu\right]$$

$$= \frac{1}{2}Cpl + \frac{1}{2}Cpu = Cp \tag{4.100}$$

and

$$Var\left[\hat{C}p\right] = Var\left[\frac{1}{2}\left(\hat{C}pl+\hat{C}pu\right)\right]$$

$$= \frac{1}{4}Var\left[\hat{C}pl\right]+\frac{1}{4}Var\left[\hat{C}pu\right]+\frac{2}{4}Cov\left[\hat{C}pl, \hat{C}pu\right]$$

$$= \left\{\frac{1}{4}+\frac{1}{4}-\frac{2}{4}\right\}\left\{\frac{1}{9}\left\{\frac{1}{n}+\left\{1-\frac{1}{n}\right\}\bar{\rho}\right\}\right\} = 0. \tag{4.101}$$

In fact, each of $\hat{Cpl}$ and $\hat{Cpu}$ is normal, each being a linear function of the normal $\bar{X}$. Yet the pair $(\hat{Cpl}, \hat{Cpu})$ is degenerate in the line $\frac{1}{2}\hat{Cpl} + \frac{1}{2}\hat{Cpu} = Cp$.

Suppose for the moment that $\bar{\rho}$ is known. Since $\hat{Cpl}$ is normal with mean Cpl and variance $\frac{1}{9}\left\{\frac{1}{n} + \left\{1 - \frac{1}{n}\right\}\bar{\rho}\right\}$, a $(1 - \alpha)$ confidence interval for the true Cpl is given by

$$\left[\hat{Cpl} - z_{\alpha/2}\sqrt{\frac{1}{9}\left\{\frac{1}{n} + \left\{1 - \frac{1}{n}\right\}\bar{\rho}\right\}},\ \hat{Cpl} + z_{\alpha/2}\sqrt{\frac{1}{9}\left\{\frac{1}{n} + \left\{1 - \frac{1}{n}\right\}\bar{\rho}\right\}}\right], \tag{4.102}$$

where $z_{\alpha/2}$ is the upper $\alpha/2$ percentile of the standard normal random variable. Similarly, a $(1 - \alpha)$ confidence interval for the true Cpu is given by

$$\left[\hat{Cpu} - z_{\alpha/2}\sqrt{\frac{1}{9}\left\{\frac{1}{n} + \left\{1 - \frac{1}{n}\right\}\bar{\rho}\right\}},\ \hat{Cpu} + z_{\alpha/2}\sqrt{\frac{1}{9}\left\{\frac{1}{n} + \left\{1 - \frac{1}{n}\right\}\bar{\rho}\right\}}\right]. \tag{4.103}$$

A joint $(1 - \alpha)$ confidence region for the true (Cpl, Cpu) is given by the line segment joining the points in (Cpl, Cpu)-space,

$$\left(\hat{Cpl} - z_{\alpha/2}\sqrt{\frac{1}{9}\left\{\frac{1}{n} + \left\{1 - \frac{1}{n}\right\}\bar{\rho}\right\}},\ \hat{Cpu} + z_{\alpha/2}\sqrt{\frac{1}{9}\left\{\frac{1}{n} + \left\{1 - \frac{1}{n}\right\}\bar{\rho}\right\}}\right)$$

$$\text{and}\quad \left(\hat{Cpl} + z_{\alpha/2}\sqrt{\frac{1}{9}\left\{\frac{1}{n} + \left\{1 - \frac{1}{n}\right\}\bar{\rho}\right\}},\ \hat{Cpu} - z_{\alpha/2}\sqrt{\frac{1}{9}\left\{\frac{1}{n} + \left\{1 - \frac{1}{n}\right\}\bar{\rho}\right\}}\right). \tag{4.104}$$

In other words, a joint $(1 - \alpha)$ confidence region for the true (Cpl, Cpu) is given by line segment in the (Cpl, Cpu) plane,

$$\gamma\left(\hat{Cpl} - z_{\alpha/2}\sqrt{\frac{1}{9}\left\{\frac{1}{n} + \left\{1 - \frac{1}{n}\right\}\bar{\rho}\right\}},\ \hat{Cpu} + z_{\alpha/2}\sqrt{\frac{1}{9}\left\{\frac{1}{n} + \left\{1 - \frac{1}{n}\right\}\bar{\rho}\right\}}\right)$$
$$+(1-\gamma)\left(\hat{Cpl} + z_{\alpha/2}\sqrt{\frac{1}{9}\left\{\frac{1}{n} + \left\{1 - \frac{1}{n}\right\}\bar{\rho}\right\}},\ \hat{Cpu} - z_{\alpha/2}\sqrt{\frac{1}{9}\left\{\frac{1}{n} + \left\{1 - \frac{1}{n}\right\}\bar{\rho}\right\}}\right)$$

(4.105)

for all real $0 \le \gamma \le 1$. Also, a joint $(1 - \alpha)$ confidence region for the true (Cpl, Cp, Cpu) is given by the line segment in (Cpl, Cp, Cpu)-space,

$$\gamma\left(\hat{C}pl - z_{\alpha/2}\sqrt{\frac{1}{9}\left\{\frac{1}{n}+\left\{1-\frac{1}{n}\right\}\bar{\rho}\right\}},\ \hat{C}p,\ \hat{C}pu + z_{\alpha/2}\sqrt{\frac{1}{9}\left\{\frac{1}{n}+\left\{1-\frac{1}{n}\right\}\bar{\rho}\right\}}\right)$$
$$+(1-\gamma)\left(\hat{C}pl + z_{\alpha/2}\sqrt{\frac{1}{9}\left\{\frac{1}{n}+\left\{1-\frac{1}{n}\right\}\bar{\rho}\right\}},\ \hat{C}p,\ \hat{C}pu - z_{\alpha/2}\sqrt{\frac{1}{9}\left\{\frac{1}{n}+\left\{1-\frac{1}{n}\right\}\bar{\rho}\right\}}\right) \tag{4.106}$$

for all real $0 \le \gamma \le 1$.

With the variance-covariance structure of the sample $\{X_i\}_n$ assumed completely known, interval estimation of the triple index (Cpl, Cp, Cpu) becomes a one-dimensional problem in the unknown parameter μ. The $(1 - \alpha)$ confidence line for the true (Cpl, Cpu) plots as a line segment perpendicular to the ray of potentiality, similar to our Figure 3.1.

4.8. Estimating (Cpl, Cp, Cpu) when μ is Known and σ is Unknown

Consider the natural estimators

$$\left(\hat{C}pl, \hat{C}p, \hat{C}pu\right) = \left(\frac{\mu - LSL}{3S},\ \frac{USL - LSL}{6S},\ \frac{USL - \mu}{3S}\right), \tag{4.107}$$

where

$$S = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2}. \tag{4.108}$$

We have

$$E\left[\hat{C}pl\right] = E\left[\frac{\mu - LSL}{3S}\right] = \frac{\mu - LSL}{3\sigma}E\left[\frac{\sigma}{S}\right]$$
$$= Cpl\ E\left[\frac{\hat{C}p}{Cp}\right] \approx Cpl\left\{1 + \frac{3}{4}\left\{\frac{1}{n}+\left\{1-\frac{1}{n}\right\}\bar{\rho}^2\right\}\right\} \tag{4.109}$$

and

$$Var\left[\hat{C}pl\right] = Var\left[\frac{\mu - LSL}{3S}\right] = \left\{\frac{\mu - LSL}{3\sigma}\right\}^2 Var\left[\frac{\sigma}{S}\right] = C^2 pl\, Var\left[\frac{\hat{C}p}{Cp}\right]$$

$$\approx C^2 pl \left\{\frac{1}{2}\left\{\frac{1}{n} + \left\{1 - \frac{1}{n}\right\}\bar{\rho}^2\right\}\right\}. \tag{4.110}$$

We also have

$$E\left[\hat{C}pu\right] = E\left[\frac{USL - \mu}{3S}\right] = \frac{USL - \mu}{3\sigma} E\left[\frac{\sigma}{S}\right]$$

$$= Cpu\, E\left[\frac{\hat{C}p}{Cp}\right] \approx Cpu \left\{1 + \frac{3}{4}\left\{\frac{1}{n} + \left\{1 - \frac{1}{n}\right\}\bar{\rho}^2\right\}\right\} \tag{4.111}$$

and

$$Var\left[\hat{C}pu\right] = Var\left[\frac{USL - \mu}{3S}\right] = \left\{\frac{USL - \mu}{3\sigma}\right\}^2 Var\left[\frac{\sigma}{S}\right] = C^2 pu\, Var\left[\frac{\hat{C}p}{Cp}\right]$$

$$\approx C^2 pu \left\{\frac{1}{2}\left\{\frac{1}{n} + \left\{1 - \frac{1}{n}\right\}\bar{\rho}^2\right\}\right\}. \tag{4.112}$$

The covariance can be gotten as

$$Cov\left[\hat{C}pl, \hat{C}pu\right] = Cov\left[\frac{\mu - LSL}{3S}, \frac{USL - \mu}{3S}\right]$$

$$= \frac{\mu - LSL}{3}\frac{USL - \mu}{3} Cov\left[\frac{1}{S}, \frac{1}{S}\right] = \frac{\mu - LSL}{3}\frac{USL - \mu}{3} Var\left[\frac{1}{S}\right]$$

$$= \frac{\mu - LSL}{3\sigma}\frac{USL - \mu}{3\sigma} Var\left[\frac{\sigma}{S}\right] = Cpl\, Cpu\, Var\left[\frac{\hat{C}p}{Cp}\right]$$

$$\approx Cpl\, Cpu \left\{\frac{1}{2}\left\{\frac{1}{n} + \left\{1 - \frac{1}{n}\right\}\bar{\rho}^2\right\}\right\}. \tag{4.113}$$

Note that $Cov\left[\hat{C}pl, \hat{C}pu\right]$ can be any real number, positive, negative, or zero. Of course, regardless of the approximations, we have

$$Corr\left[\hat{C}pl, \hat{C}pu\right] = \frac{Cov\left[\hat{C}pl, \hat{C}pu\right]}{\sqrt{Var\left[\hat{C}pl\right]}\sqrt{Var\left[\hat{C}pu\right]}} = \pm 1, \tag{4.114}$$

provided μ does not fall at either LSL or USL. If μ falls within the specification limits, then $Cov\left[\hat{C}pl, \hat{C}pu\right]$ is positive one. If μ falls outside, then it is negative one. There are problems at the two poles $\mu = LSL$ and $\mu = USL$, at which $\hat{C}pl$ or $\hat{C}pu$ is identically zero, and so $Var\left[\hat{C}pl\right]$ or $Var\left[\hat{C}pu\right]$ is zero. Note further that

$$E\left[\hat{C}p\right] = E\left[\frac{1}{2}\left(\hat{C}pl + \hat{C}pl\right)\right]$$

$$\approx \frac{1}{2}Cpl\left\{1 + \frac{3}{4}\left\{\frac{1}{n} + \left\{1 - \frac{1}{n}\right\}\bar{\rho}^2\right\}\right\} + \frac{1}{2}Cpu\left\{1 + \frac{3}{4}\left\{\frac{1}{n} + \left\{1 - \frac{1}{n}\right\}\bar{\rho}^2\right\}\right\}$$

$$= \frac{1}{2}(Cpl + Cpu)\left\{1 + \frac{3}{4}\left\{\frac{1}{n} + \left\{1 - \frac{1}{n}\right\}\bar{\rho}^2\right\}\right\}$$

$$= Cp\left\{1 + \frac{3}{4}\left\{\frac{1}{n} + \left\{1 - \frac{1}{n}\right\}\bar{\rho}^2\right\}\right\} \tag{4.115}$$

and

$$Var\left[\hat{C}p\right] = Var\left[\frac{1}{2}\left(\hat{C}pl + \hat{C}pl\right)\right]$$

$$= \frac{1}{4}Var\left[\hat{C}pl\right] + \frac{1}{4}Var\left[\hat{C}pu\right] + \frac{2}{4}Cov\left[\hat{C}pl, \hat{C}pu\right]$$

$$\approx \left(\frac{1}{4}C^2pl + \frac{1}{4}C^2pu + \frac{2}{4}Cpl\,Cpu\right)\left\{\frac{1}{2}\left\{\frac{1}{n} + \left\{1 - \frac{1}{n}\right\}\bar{\rho}^2\right\}\right\}$$

$$= \left\{\frac{1}{2}(Cpl + Cpu)\right\}^2\left\{\frac{1}{2}\left\{\frac{1}{n} + \left\{1 - \frac{1}{n}\right\}\bar{\rho}^2\right\}\right\}$$

$$= C^2p\left\{\frac{1}{2}\left\{\frac{1}{n} + \left\{1 - \frac{1}{n}\right\}\bar{\rho}^2\right\}\right\}, \tag{4.116}$$

which are consistent with equations (4.62) and (4.63).

4.9. Estimating (*Cpl*, *Cp*, *Cpu*) when Both μ and σ are Unknown

We now consider the natural estimators

$$\left(\hat{C}pl, \hat{C}p, \hat{C}pu\right) = \left(\frac{\bar{X} - LSL}{3S}, \ \frac{USL - LSL}{6S}, \ \frac{USL - \bar{X}}{3S}\right), \tag{4.117}$$

where

$$(\bar{X}, S) = \left(\frac{1}{n} \sum_{i=1}^{n} X_i, \ \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2} \right).$$

We have previously obtained approximations for $E[\hat{C}p/Cp]$ and $Var[\hat{C}p/Cp]$, given in equations (4.82) and (4.83). It follows that

$$E[\hat{C}p] \approx \hat{E}[\hat{C}p] = \frac{Cp}{\sqrt{1-\bar{\rho}}} \left\{ 1 + \frac{3}{4} \frac{trace\mathbf{CRCR}}{trace\mathbf{CR}\ trace\mathbf{CR}} \right\} \tag{4.118}$$

and

$$Var[\hat{C}p] \approx \hat{V}ar[\hat{C}p] = \frac{C^2 p}{2(1-\bar{\rho})} \frac{trace\mathbf{CRCR}}{trace\mathbf{CR}\ trace\mathbf{CR}}. \tag{4.119}$$

Because of the dependence of $(\bar{X}, S)$, the first two moments of $\hat{C}pl = (\bar{X} - LSL)/3S$ and $\hat{C}pu = (USL - \bar{X})/3S$ are much more difficult to approximate. We have

$$\begin{aligned}
E[\hat{C}pl] &= E\left[\frac{\bar{X} - LSL}{3S}\right] = \frac{1}{3} E\left[\frac{\bar{X}}{S}\right] - \frac{LSL}{3} E\left[\frac{1}{S}\right] \\
&= \frac{1}{3}\left\{ E[\bar{X}] E\left[\frac{1}{S}\right] + Cov\left[\bar{X}, \frac{1}{S}\right] \right\} - \frac{LSL}{3} E\left[\frac{1}{S}\right] \\
&= \frac{1}{3\sigma}\left\{ E[\bar{X}] E\left[\frac{\sigma}{S}\right] + Cov\left[\bar{X}, \frac{\sigma}{S}\right] \right\} - \frac{LSL}{3\sigma} E\left[\frac{\sigma}{S}\right] \\
&= \frac{E[\bar{X}] - LSL}{3\sigma} E\left[\frac{\sigma}{S}\right] + \frac{1}{3\sigma} Cov\left[\bar{X}, \frac{\sigma}{S}\right] \\
&= Cpl\ E\left[\frac{\hat{C}p}{Cp}\right] + \frac{1}{3} Cov\left[\bar{X}, \frac{1}{S}\right] \\
&= Cpl\ E\left[\frac{\hat{C}p}{Cp}\right] + \frac{1}{3} \sqrt{Var[\bar{X}]} \sqrt{Var[1/S]}\ Corr\left[\bar{X}, \frac{1}{S}\right] \\
&\approx Cpl \left\{ \frac{1}{\sqrt{1-\bar{\rho}}} \left\{ 1 + \frac{3}{4} \frac{trace\mathbf{CRCR}}{trace\mathbf{CR}\ trace\mathbf{CR}} \right\} \right\}
\end{aligned}$$

$$+\frac{1}{3}\sqrt{\sigma^2\left\{\frac{1}{n}+\left\{1-\frac{1}{n}\right\}\bar{\rho}\right\}}\sqrt{\frac{1}{2\sigma^2(1-\bar{\rho})}\frac{trace\mathbf{CRCR}}{trace\mathbf{CR}\;trace\mathbf{CR}}}\,Corr\left[\bar{X},\frac{1}{S}\right]$$

$$=Cpl\left\{\frac{1}{\sqrt{1-\bar{\rho}}}\left\{1+\frac{3}{4}\frac{trace\mathbf{CRCR}}{trace\mathbf{CR}\;trace\mathbf{CR}}\right\}\right\}$$

$$+\frac{1}{3}\sqrt{\frac{1}{n}+\left\{1-\frac{1}{n}\right\}\bar{\rho}}\sqrt{\frac{1}{2(1-\bar{\rho})}\frac{trace\mathbf{CRCR}}{trace\mathbf{CR}\;trace\mathbf{CR}}}\,Corr\left[\bar{X},\frac{1}{S}\right], \tag{4.120}$$

where the approximations are gotten from equations (4.82), (4.83), and (4.93). In a similar manner, we have

$$E\left[\hat{C}pu\right]=E\left[\frac{USL-\bar{X}}{3S}\right]=Cpu\,E\left[\frac{\hat{C}p}{Cp}\right]-\frac{1}{3}Cov\left[\bar{X},\frac{1}{S}\right]$$

$$=Cpu\,E\left[\frac{\hat{C}p}{Cp}\right]-\frac{1}{3}\sqrt{Var\left[\bar{X}\right]}\sqrt{Var\left[1/S\right]}\,Corr\left[\bar{X},\frac{1}{S}\right]$$

$$\approx Cpu\left\{\frac{1}{\sqrt{1-\bar{\rho}}}\left\{1+\frac{3}{4}\frac{trace\mathbf{CRCR}}{trace\mathbf{CR}\;trace\mathbf{CR}}\right\}\right\}$$

$$-\frac{1}{3}\sqrt{\sigma^2\left\{\frac{1}{n}+\left\{1-\frac{1}{n}\right\}\bar{\rho}\right\}}\sqrt{\frac{1}{2\sigma^2(1-\bar{\rho})}\frac{trace\mathbf{CRCR}}{trace\mathbf{CR}\;trace\mathbf{CR}}}\,Corr\left[\bar{X},\frac{1}{S}\right]$$

$$=Cpu\left\{\frac{1}{\sqrt{1-\bar{\rho}}}\left\{1+\frac{3}{4}\frac{trace\mathbf{CRCR}}{trace\mathbf{CR}\;trace\mathbf{CR}}\right\}\right\}$$

$$-\frac{1}{3}\sqrt{\frac{1}{n}+\left\{1-\frac{1}{n}\right\}\bar{\rho}}\sqrt{\frac{1}{2(1-\bar{\rho})}\frac{trace\mathbf{CRCR}}{trace\mathbf{CR}\;trace\mathbf{CR}}}\,Corr\left[\bar{X},\frac{1}{S}\right]. \tag{4.121}$$

Note that we have

$$\frac{1}{2}E\left[\hat{C}pl\right]+\frac{1}{2}E\left[\hat{C}pu\right]=E\left[\hat{C}p\right] \quad\text{and}\quad \frac{1}{2}\hat{E}\left[\hat{C}pl\right]+\frac{1}{2}\hat{E}\left[\hat{C}pu\right]=\hat{E}\left[\hat{C}p\right], \tag{4.122}$$

as they should, but $Corr\left[\bar{X},\frac{1}{S}\right]$ is still a missing piece to the puzzle.

Since $Corr\left[\bar{X},\frac{1}{S}\right]$ is not amenable to Taylor series approximation in this very complex case of autocorrelated observations, we must turn to the problem of estimating the natural parameters in the ARMA(p, q) model. Now Box and Jenkins (1993) favor the maximum likelihood (ML) criterion for choosing coefficient estimates, since the likelihood function from which ML estimates are derived reflects all useful information in the data about the parameters. However, finding exact ML estimates can be computationally burdensome, and least-squares (LS) estimates are an alternative. If the random shocks are normally distributed, LS estimates provide exactly, or very nearly, ML estimates.

LS estimates are those which give the smallest sum of squared residuals SSR. Linear least-squares (LLS) may be used to estimate only pure AR models without multiplicative seasonal terms, but all other models require a nonlinear least-squares (NLS) method. One NLS method is the grid-search procedure, where the coefficient is assigned a series of admissible values and an SSR is found for each combination of these values. The combination of coefficients with the smallest SSR is chosen as the set of LS estimates. This method is not often used because of the great time involved in evaluating the sum of squared residuals for the many combinations of coefficient estimates.

The most commonly used NLS method is Marquandt's compromise. Marquandt's method is called a compromise because it combines the best features of Gauss-Newton linearization and the gradient method, also known as the steepest-descent method. The practical advantage of the Gauss-Newton method is that it tends to converge rapidly to the least-squares (LS) estimates, if it converges. The disadvantage is that it may not converge at all. The practical advantage of the gradient method is that, in theory, it will converge to the LS estimates. However, it may converge so slowly that it becomes impractical to use.

Marquandt's compromise combines the best of these two approaches. Except in rare cases, it converges quickly to the LS estimates.

In addition to its feature of reflecting all information in the sample, maximum likelihood estimates possess the attractive property of asymptotic normality. If the sample size n is sufficiently large, the distribution of the vector of maximum likelihood estimates $\hat{\theta}$ can be well approximated by a multivariate normal distribution,

$$\hat{\theta} \sim N\left(\theta_0, \ \frac{1}{n}\mathbf{Inf}^{-1}(\theta)\right), \tag{4.123}$$

where θ_0 denotes the true parameter vector and $\mathbf{Inf}(\theta)$ is the information matrix. An estimate of the information matrix is

$$\hat{\mathbf{Inf}} = -\frac{1}{n}\frac{\partial^2 \log L(\theta)}{\partial\theta\,\partial\theta'}\Bigg|_{\theta=\hat{\theta}}, \tag{4.124}$$

where $\log L(\theta)$ denotes the log likelihood function

$$\log L(\theta) = \sum_{i=1}^{n} \log f_{X_i|\Omega_{i-1}}\left(x_i|\Omega_{i-1}, \theta\right), \tag{4.125}$$

and Ω_{i-1} denotes the history of observations on X obtained through time $(i - 1)$. The matrix of second derivatives of the log likelihood is often calculated numerically. Substituting equation (4.124) into equation (4.123), the terms involving the sample size n cancel so that

$$E\left(\hat{\theta}-\theta_0\right)\left(\hat{\theta}-\theta_0\right)' \approx \left[-\frac{\partial^2 \log L(\theta)}{\partial\theta\,\partial\theta'}\Bigg|_{\theta=\hat{\theta}}\right]^{-1} = \left[n\,\hat{\mathbf{Inf}}\right]^{-1}. \tag{4.126}$$

In this way, an approximate $(1 - \alpha)$ confidence ellipsoid for the true parameter vector θ_0 can be gotten as

$$n\left(\hat{\theta}-\theta_0\right)'\,\hat{\mathbf{Inf}}\,\left(\hat{\theta}-\theta_0\right) \le \chi^2_{p+q+2,\,1-\alpha}, \tag{4.127}$$

where $\chi^2_{p+q+2,1-\alpha}$ is the lower (1 - α) percentile of the chi-squared random variable with degrees of freedom (p + q + 2) representing the dimension of the parameter vector θ_0.

Suppose the process follows an AR(1), that is, $(X_t - \mu) - \phi(X_{t-1} - \mu) = a_t$, where the a_t are *iid* $N(0, \sigma_a^2)$ and $-1 < \phi < 1$. The variance of the process characteristic X_t is given by $Var[X_t] = \sigma^2 = \sigma_a^2 / (1 - \phi^2)$. Given the maximum likelihood estimates $(\hat{\mu}, \hat{\sigma}_a, \hat{\phi})$ of the natural population parameters (μ, σ_a, ϕ), we can determine the maximum likelihood estimates of the reduced parameters (μ, σ_X) by

$$(\hat{\mu}, \hat{\sigma}_X) = \left(\hat{\mu}, \quad \hat{\sigma}_a \sqrt{\frac{1}{1 - \hat{\phi}^2}} \right), \tag{4.128}$$

which, in turn, determine the maximum likelihood estimates of (Cpl, Cp, Cpu),

$$(\hat{C}pl, \hat{C}p, \hat{C}pu) = \left(\frac{\hat{\mu} - LSL}{3\hat{\sigma}_X}, \quad \frac{USL - LSL}{6\hat{\sigma}_X}, \quad \frac{USL - \hat{\mu}}{3\hat{\sigma}_X} \right)$$

$$= \left(\frac{\hat{\mu} - LSL}{3\hat{\sigma}_a \sqrt{\frac{1}{1-\hat{\phi}^2}}}, \quad \frac{USL - LSL}{6\hat{\sigma}_a \sqrt{\frac{1}{1-\hat{\phi}^2}}}, \quad \frac{USL - \hat{\mu}}{3\hat{\sigma}_a \sqrt{\frac{1}{1-\hat{\phi}^2}}} \right). \tag{4.129}$$

Let **G** be the gradient matrix

$$\mathbf{G} = \begin{bmatrix} \frac{\partial Cpl}{\partial \mu} & \frac{\partial Cpl}{\partial \sigma_a} & \frac{\partial Cpl}{\partial \phi} \\ \frac{\partial Cpu}{\partial \mu} & \frac{\partial Cpu}{\partial \sigma_a} & \frac{\partial Cpu}{\partial \phi} \end{bmatrix}_{2\times 3}$$

$$= \begin{bmatrix} \frac{\sqrt{1-\phi^2}}{3\sigma_a} & \frac{-(\mu - LSL)\sqrt{1-\phi^2}}{3\sigma_a^2} & \frac{-(\mu - LSL)\phi\sqrt{1-\phi^2}}{3\sigma_a} \\ \frac{-\sqrt{1-\phi^2}}{3\sigma_a} & \frac{-(USL-\mu)\sqrt{1-\phi^2}}{3\sigma_a^2} & \frac{-(USL-\mu)\phi\sqrt{1-\phi^2}}{3\sigma_a} \end{bmatrix}. \tag{4.130}$$

We have, by Theil (1971), page 383,

$$\left(\hat{C}pl, \hat{C}pu\right) \sim N_2\left((Cpl, Cpu),\ \frac{1}{n}\mathbf{G}\mathbf{Inf}^{-1}\mathbf{G}'\right), \tag{4.131}$$

approximately for large n. From this, a $(1 - \alpha)$ joint confidence ellipse for the true (Cpl, Cpu) is found to be

$$n(\hat{\mathbf{c}}-\mathbf{c})'\left(\hat{\mathbf{G}}\hat{\mathbf{I}}\mathbf{nf}^{-1}\hat{\mathbf{G}}'\right)^{-1}(\hat{\mathbf{c}}-\mathbf{c}) \le \chi^2_{2,1-\alpha}, \tag{4.132}$$

where $\hat{\mathbf{c}} = \left(\hat{C}pl, \hat{C}pu\right)'$, $\mathbf{c} = (Cpl, Cpu)'$, $\chi^2_{2,1-\alpha}$ is the lower $(1 - \alpha)$ percentile of the chi-squared random variable with two degrees of freedom, $\hat{\mathbf{I}}\mathbf{nf}^{-1}$ is the inverse of the numerically estimated information matrix, and $\hat{\mathbf{G}}$ is the estimated gradient matrix

$$\hat{\mathbf{G}} = \begin{bmatrix} \dfrac{\sqrt{1-\hat{\phi}^2}}{3\hat{\sigma}_a} & \dfrac{-(\hat{\mu}-LSL)\sqrt{1-\hat{\phi}^2}}{3\hat{\sigma}_a^2} & \dfrac{-(\hat{\mu}-LSL)\hat{\phi}\sqrt{1-\hat{\phi}^2}}{3\hat{\sigma}_a} \\ \dfrac{-\sqrt{1-\hat{\phi}^2}}{3\hat{\sigma}_a} & \dfrac{-(USL-\hat{\mu})\sqrt{1-\hat{\phi}^2}}{3\hat{\sigma}_a^2} & \dfrac{-(USL-\hat{\mu})\hat{\phi}\sqrt{1-\hat{\phi}^2}}{3\hat{\sigma}_a} \end{bmatrix}. \tag{4.133}$$

This procedure involves much approximation and it would seem foolish to attempt such with small samples. Potential inaccuracies are exacerbated in higher order ARMA(p, q) models, where the ψ parameters may decay slowly. Box and Jenkins (1993) suggest a sample size n of 50 or larger.

Fortunately, we are seeking an estimate of $(\mu, \sigma_X) = \left(\mu,\ \sigma_a\sqrt{\sum_{i=1}^{\infty}\psi_i^2}\right)$ and not strictly the constituent parameters $(\mu, \sigma_a, \psi_1, \psi_2, \cdots)$. Now since the random variables

$$(\bar{X}, S) = \left(\frac{1}{n}\sum_{i=1}^{n}X_i,\ \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2}\right)$$

approach

$$(\mu, \sigma_X) = \left(\mu, \quad \sigma_a \sqrt{\sum_{i=1}^{\infty} \psi_i^2}\right)$$

both in mean square and in probability, we know that the estimators

$$\left(\hat{Cpl}, \hat{Cp}, \hat{Cpu}\right) = \left(\frac{\overline{X} - LSL}{3S}, \quad \frac{USL - LSL}{6S}, \quad \frac{USL - \overline{X}}{3S}\right)$$

approach the true (*Cpl*, *Cp*, *Cpu*) both in mean square and in probability. Furthermore, we see from Table 4.9 that $E\left[\hat{Cp}/Cp\right]$ is less than 1.10 for n at least 40 and ϕ no more than 0.6 in the AR(1) model, suggesting that the approach to the true parameters is sufficiently fast, at least if the autocorrelation is not too high.

CHAPTER 5. SUMMARY AND CONCLUSIONS

Gunter (1989) gives the example of the well-meaning manager who required that all suppliers providing prototype parts for his department meet minimum *Cpk* standards, despite the fact that no processes for producing the parts yet existed and that the total number of parts to be purchased was less than two dozen. A second example was that of the supplier who, in order to meet the *Cpk* requirements of a large customer, made sure that all parts collected for the *Cpk* measurements were made by the most skilled operator on the best machine. Gunter's final example is that of the plant that prided itself on its high *Cpk* despite the fact that control charts showed the processes to be mostly out-of-control.

Examples such as the three above are often cited by critics of the process capability indices as instances of the weaknesses inherent in the indices. Of course, these scenarios do *not* show drawbacks in the indices as much as they point to the dangers of ignoring the probabilistic assumptions necessary for the reasonable use of these indices. If the process capability indices had not been invented, and the natural parameters (μ, σ) were used in each of the above examples, one would be standing on the same shaky ground. This is because estimation of the natural parameters (μ, σ) is equivalent to estimation of the triple process capability index (*Cpl*, *Cp*, *Cpu*) whenever *LSL* and *USL* are known and the process X has a marginal normal probability density function (pdf) with mean μ and standard deviation σ, independent of t.

The equivalence of the two parameterizations was demonstrated in Chapter 1. Once this equivalence is understood, misuse of the process capability indices is seen in the broader context. This can include anything from a simple failure to account for the sampling variability of the estimators, to misspecification of the population generating the random characteristics. Yet emphatically, the indices *themselves* are not the source of these failures.

Following the literature review of Chapter 2, our third chapter addressed estimation of the three process capability indices (*Cpl*, *Cp*, *Cpu*) for independent, identically distributed normal characteristics. We discussed the apparent normality of observed data. When to expect it and when not to. We gave the common point estimators of (*Cpl*, *Cp*, *Cpu*), including maximum likelihood (ML), uniformly minimum variance unbiased (UMVU), and natural moment-based estimators. We continued with classical interval estimation, presenting a method for determining a joint confidence interval for the true triple index (*Cpl*, *Cp*, *Cpu*) which is, both conceptually and computationally, less obtuse than any method previously published. For our effort, we were rewarded with both point and interval estimators of the proportion π_0 of product outside specification, a parameter which many experts feel to be the *raison d'être* of process capability analysis. Also in Chapter 3, we investigated the performance of Taylor series approximations to particular means and variances, as a prelude to their use in Chapter 4, where they became indispensible rather than merely interesting.

We began Chapter 4 with a brief, elementary excursion into linear stochastic differential equations. This provided the necessary insight into the phenomenon of autocorrelation and its effect on the sample variance. We proved the existence of a lower bound on the mean of the random variable $\hat{C}p/Cp$ for autocorrelated data. We then investigated the common point and interval estimators of (*Cpl*, *Cp*, *Cpu*) under the stationary normal ARMA(p, q) model, thereby broadening the realm of applicability of these process capability indices. This investigation necessitated using Taylor-series-based approximations for the mean and variance of a random variable. These approximations, while analytically intensive in their derivation, ultimately proved satisfying, both practically by their acceptable relative errors, and aesthetically in their artistic proportions. We conclude that autocorrelation seriously compromises naive estimators of (*Cpl*, *Cp*, *Cpu*) for small samples and large population correlation parameters. When sampling from autocorrelated models, *ceteris paribus*, more data is better than less data.

Issues not addressed include the sensitivity of (*Cpl*, *Cp*, *Cpu*) to non-normality, and the presence of systematic measurement error in the sampled data. While these important topics have been approached by researchers (see Chapter 2), remedies have been less than satisfactory and more work should be done.

We note that our focus on the triple index (*Cpl*, *Cp*, *Cpu*) is at odds with the current practice, which tends to emphasize the index pair (*Cp*, *Cpk*). We believe our approach to be superior in that it maintains an equivalence with (μ, σ), given *LSL* and *USL*, while sacrificing nothing. In fact, we are convinced that the intensive academic attention given the *Cpk* index has resulted in criticisms of the (*Cpl*, *Cp*, *Cpu*) indices, from some quarters, that they hardly deserve.

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APPENDIX A. DERIVATION OF AN IMPORTANT MEAN

We have two expressions for $E\left[\hat{C}pk/Cpk\right]$,

$$E\left[\frac{\hat{C}pk}{Cpk}\right]=\frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}}\left\{\Phi+\frac{USL-\mu}{\mu-LSL}(1-\Phi)-2\frac{\phi\sigma/\sqrt{n}}{\mu-LSL}\right\}\quad \text{for}\quad \mu\le m$$

and

$$E\left[\frac{\hat{C}pk}{Cpk}\right]=\frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}}\left\{\frac{\mu-LSL}{USL-\mu}\Phi+(1-\Phi)-2\frac{\phi\sigma/\sqrt{n}}{USL-\mu}\right\}\quad \text{for}\quad \mu> m,$$

(A.1)

where ϕ and Φ are the standard normal pdf and cdf evaluated at $\frac{m-\mu}{\sigma/\sqrt{n}}$.

We can reparameterize these two expressions into a single expression for $E\left[\hat{C}pk/Cpk\right]$. Let $\Delta=\frac{USL-LSL}{\sigma}$ be the length of the specification interval in process standard deviations and let $\delta=\left|\frac{m-\mu}{\sigma}\right|$ be the unsigned "offset" distance between the process mean and the specification interval midpoint in process standard deviations.

First, consider that for $\mu\le m=(LSL+USL)/2$, $\Delta=\frac{USL-LSL}{\sigma}$, and $\delta=\left|\frac{m-\mu}{\sigma}\right|$, we have $\frac{USL-\mu}{\sigma}=\frac{1}{2}\Delta+\delta$, $\frac{\mu-LSL}{\sigma}=\frac{1}{2}\Delta-\delta$, and $\frac{USL-\mu}{\mu-LSL}=\frac{\frac{1}{2}\Delta+\delta}{\frac{1}{2}\Delta-\delta}$. Substituting into the first line of equation (A.1) gives

$$E\left[\frac{\hat{C}pk}{Cpk}\right]=\frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}}\left\{\Phi+\frac{USL-\mu}{\mu-LSL}(1-\Phi)-2\frac{\phi\sigma/\sqrt{n}}{\mu-LSL}\right\}$$

$$=\frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}}\left\{\Phi+\frac{\frac{1}{2}\Delta+\delta}{\frac{1}{2}\Delta-\delta}(1-\Phi)-2\frac{\phi/\sqrt{n}}{\frac{1}{2}\Delta-\delta}\right\}$$

$$=\frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}}\left\{\Phi+\frac{\frac{1}{2}\Delta+\delta}{\frac{1}{2}\Delta-\delta}-\frac{\frac{1}{2}\Delta+\delta}{\frac{1}{2}\Delta-\delta}\Phi-2\frac{\phi/\sqrt{n}}{\frac{1}{2}\Delta-\delta}\right\}$$

$$=\frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}}\left\{\frac{\frac{1}{2}\Delta-\delta}{\frac{1}{2}\Delta-\delta}\Phi+\frac{\frac{1}{2}\Delta+\delta}{\frac{1}{2}\Delta-\delta}-\frac{\frac{1}{2}\Delta+\delta}{\frac{1}{2}\Delta-\delta}\Phi-2\frac{\phi/\sqrt{n}}{\frac{1}{2}\Delta-\delta}\right\}$$

$$=\frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}}\left\{\frac{\frac{1}{2}\Delta+\delta}{\frac{1}{2}\Delta-\delta}+\left\{\frac{\frac{1}{2}\Delta-\delta}{\frac{1}{2}\Delta-\delta}-\frac{\frac{1}{2}\Delta+\delta}{\frac{1}{2}\Delta-\delta}\right\}\Phi-2\frac{\phi/\sqrt{n}}{\frac{1}{2}\Delta-\delta}\right\}$$

$$=\frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}}\left\{1+\frac{2\delta}{\frac{1}{2}\Delta-\delta}-\frac{2\delta}{\frac{1}{2}\Delta-\delta}\Phi-2\frac{\phi/\sqrt{n}}{\frac{1}{2}\Delta-\delta}\right\}$$

$$=\frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}}\left\{1+\frac{2\delta(1-\Phi)-2\phi/\sqrt{n}}{\frac{1}{2}\Delta-\delta}\right\}$$

$$=\frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}}\left\{1+4\left\{\frac{\delta(1-\Phi)-\phi/\sqrt{n}}{\Delta-2\delta}\right\}\right\}, \qquad (A.2)$$

where ϕ and Φ are the standard normal pdf and cdf evaluated at $\frac{m-\mu}{\sigma/\sqrt{n}}=\left|\frac{m-\mu}{\sigma/\sqrt{n}}\right|=\delta\sqrt{n}$.

On the other hand, for $\mu>m=(LSL+USL)/2$, $\Delta=\frac{USL-LSL}{\sigma}$, and $\delta=\left|\frac{m-\mu}{\sigma}\right|$, we have $\frac{\mu-LSL}{\sigma}=\frac{1}{2}\Delta+\delta$, $\frac{USL-\mu}{\sigma}=\frac{1}{2}\Delta-\delta$, and $\frac{\mu-LSL}{USL-\mu}=\frac{\frac{1}{2}\Delta+\delta}{\frac{1}{2}\Delta-\delta}$. Substituting into the second line of equation (A.1) gives

$$E\left[\frac{\hat{Cpk}}{Cpk}\right]=\frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}}\left\{\frac{\mu-LSL}{USL-\mu}\Phi+(1-\Phi)-2\frac{\phi\sigma/\sqrt{n}}{USL-\mu}\right\}$$

$$=\frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}}\left\{\frac{\frac{1}{2}\Delta+\delta}{\frac{1}{2}\Delta-\delta}\Phi+(1-\Phi)-\frac{2\phi/\sqrt{n}}{\frac{1}{2}\Delta-\delta}\right\}$$

$$=\frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}}\left\{1+\frac{\frac{1}{2}\Delta+\delta}{\frac{1}{2}\Delta-\delta}\Phi-\Phi-\frac{2\phi/\sqrt{n}}{\frac{1}{2}\Delta-\delta}\right\}$$

$$=\frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}}\left\{1+\frac{2\delta}{\frac{1}{2}\Delta-\delta}\Phi-\frac{2\phi/\sqrt{n}}{\frac{1}{2}\Delta-\delta}\right\}$$

$$=\frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}}\left\{1+\frac{2\delta\Phi-2\phi/\sqrt{n}}{\frac{1}{2}\Delta-\delta}\right\}$$

$$= \frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}}\left\{1+4\left\{\frac{\delta\Phi-\phi/\sqrt{n}}{\Delta-2\delta}\right\}\right\}, \tag{A.3}$$

where ϕ and Φ are the standard normal pdf and cdf evaluated at $\frac{m-\mu}{\sigma/\sqrt{n}}$. But for $\mu > m$, we have $\frac{m-\mu}{\sigma/\sqrt{n}} = -\left|\frac{m-\mu}{\sigma/\sqrt{n}}\right| = -\delta\sqrt{n}$, and so

$$\Phi\left[\frac{m-\mu}{\sigma/\sqrt{n}}\right] = \Phi\left[-\delta\sqrt{n}\right] = 1-\Phi\left[\delta\sqrt{n}\right]$$

and

$$\phi\left[\frac{m-\mu}{\sigma/\sqrt{n}}\right] = \phi\left[-\delta\sqrt{n}\right] = \phi\left[\delta\sqrt{n}\right].$$

We can therefore express equation (A.3) as

$$E\left[\frac{\hat{C}pk}{Cpk}\right] = \frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}}\left\{1+4\left\{\frac{\delta(1-\Phi)-\phi/\sqrt{n}}{\Delta-2\delta}\right\}\right\} \tag{A.4}$$

for $\mu > m$, where ϕ and Φ are the standard normal pdf and cdf evaluated at $\left|\frac{m-\mu}{\sigma/\sqrt{n}}\right| = \delta\sqrt{n}$.

This is identical to the earlier equation (A.2) for $E\left[\hat{C}pk/Cpk\right]$ in the region of the parameter space $\mu \le m$. Therefore, we have the single expression for $E\left[\hat{C}pk/Cpk\right]$,

$$E\left[\frac{\hat{C}pk}{Cpk}\right] = \frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}\sqrt{\frac{n-1}{2}}\left\{1+4\left\{\frac{\delta(1-\Phi)-\phi/\sqrt{n}}{\Delta-2\delta}\right\}\right\}, \tag{A.5}$$

where ϕ and Φ are the standard normal pdf and cdf evaluated at $\left|\frac{m-\mu}{\sigma/\sqrt{n}}\right| = \left|\frac{m-\mu}{\sigma}\right|\sqrt{n} = \delta\sqrt{n}$ and $\Delta = \frac{USL-LSL}{\sigma}$.

APPENDIX B. MOMENTS OF THE MULTIVARIATE NORMAL

Let (X_1, X_2, X_3, X_4) be multivariate normal with equal means μ, equal variances σ^2, and covariances σ_{12}, σ_{13}, σ_{14}, σ_{23}, σ_{24}, and σ_{34}, not necessarily equal to zero. We wish to demonstrate the following moments by derivation from the multivariate normal moment generating function,

$$E[X_1] = \mu$$

$$E\left[X_1^2\right] = \sigma^2 + \mu^2$$

$$E[X_1 X_2] = \sigma_{12} + \mu^2$$

$$E\left[X_1^3\right] = 3\mu\sigma^2 + \mu^3$$

$$E\left[X_1^2 X_2\right] = \mu\left(\sigma^2 + 2\sigma_{12}\right) + \mu^3$$

$$E[X_1 X_2 X_3] = \mu(\sigma_{12} + \sigma_{13} + \sigma_{23}) + \mu^3$$

$$E\left[X_1^4\right] = 3\sigma^4 + 6\mu^2\sigma^2 + \mu^4$$

$$E\left[X_1^3 X_2\right] = 3\sigma_{12}\left(\sigma^2 + \mu^2\right) + 3\mu^2\sigma^2 + \mu^4$$

$$E\left[X_1^2 X_2^2\right] = \sigma^4 + 2\sigma_{12}^2 + 2\mu^2\left(\sigma^2 + 2\sigma_{12}\right) + \mu^4$$

$$E\left[X_1^2 X_2 X_3\right] = 2\sigma_{12}\sigma_{13} + \sigma_{23}\left(\sigma^2 + \mu^2\right) + 2\mu^2(\sigma_{12} + \sigma_{13}) + \mu^2\sigma^2 + \mu^4$$

$$E[X_1 X_2 X_3 X_4] = \sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{23}\sigma_{14}$$
$$+\mu^2(\sigma_{12} + \sigma_{13} + \sigma_{14} + \sigma_{23} + \sigma_{24} + \sigma_{34}) + \mu^4.$$

It will be convenient to also have these moments expressed in terms of the pairwise correlation coefficients ρ_{12}, ρ_{13}, ρ_{14}, ρ_{23}, ρ_{24}, and ρ_{34},

$$E[X_1] = \mu$$

$$E\left[X_1^2\right] = \sigma^2 + \mu^2$$

$$E[X_1 X_2] = \sigma^2\rho_{12} + \mu^2$$

$$E\left[X_1^3\right] = 3\mu\sigma^2 + \mu^3$$

$$E\left[X_1^2 X_2\right] = \mu\sigma^2(1 + 2\rho_{12}) + \mu^3$$

$$E[X_1 X_2 X_3] = \mu\sigma^2(\rho_{12} + \rho_{13} + \rho_{23}) + \mu^3$$

$$E\left[X_1^4\right] = 3\sigma^4 + 6\mu^2\sigma^2 + \mu^4$$

$$E\left[X_1^3 X_2\right] = 3\sigma^2\rho_{12}\left(\sigma^2 + \mu^2\right) + 3\mu^2\sigma^2 + \mu^4$$

$$E\left[X_1^2 X_2^2\right] = \sigma^4\left(1 + 2\rho_{12}^2\right) + 2\mu^2\sigma^2(1 + 2\rho_{12}) + \mu^4$$

$$E\left[X_1^2 X_2 X_3\right] = 2\sigma^4\rho_{12}\rho_{13} + \sigma^2\rho_{23}\left(\sigma^2 + \mu^2\right) + 2\mu^2\sigma^2(\rho_{12} + \rho_{13}) + \mu^2\sigma^2 + \mu^4$$

$$E[X_1 X_2 X_3 X_4] = \sigma^4(\rho_{12}\rho_{34} + \rho_{13}\rho_{24} + \rho_{23}\rho_{14})$$
$$+\mu^2\sigma^2(\rho_{12} + \rho_{13} + \rho_{14} + \rho_{23} + \rho_{24} + \rho_{34}) + \mu^4.$$

Let (X_1, X_2, X_3, X_4) be multivariate normal with equal means μ, equal variances σ^2, and covariances $\sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{23}, \sigma_{24}$, and σ_{34}, not necessarily equal. The moment generating function is given by

$$m(t_1, t_2, t_3, t_4) = \exp\left\{\mu(t_1 + t_2 + t_3 + t_4) + \tfrac{1}{2}\left[\sigma^2\left(t_1^2 + t_2^2 + t_3^2 + t_4^2\right)\right.\right.$$
$$\left.\left.+2\sigma_{12}t_1t_2 + 2\sigma_{13}t_1t_3 + 2\sigma_{14}t_1t_4 + 2\sigma_{23}t_2t_3 + 2\sigma_{24}t_2t_4 + 2\sigma_{34}t_3t_4\right]\right\}. \tag{B.1}$$

Note that $m(0,0,0,0) = 1$.

The first partial derivative of m with respect to t_1 is given by

$$\frac{\partial m}{\partial t_1} = m(t_1, t_2, t_3, t_4) \times \left[\mu + \sigma^2 t_1 + \sigma_{12}t_2 + \sigma_{13}t_3 + \sigma_{14}t_4\right], \tag{B.2}$$

yielding

$$\frac{\partial m}{\partial t_1}(0,0,0,0) = E[X_1] = \mu. \tag{B.3}$$

The second partial derivative of m with respect to (t_1, t_2) is given by

$$\frac{\partial^2 m}{\partial t_1 \partial t_2} = \sigma_{12} m(t_1, t_2, t_3, t_4)$$

$$+\left[\mu+\sigma^2 t_1+\sigma_{12}t_2+\sigma_{13}t_3+\sigma_{14}t_4\right]$$

$$\times m(t_1,t_2,t_3,t_4)\left[\mu+\sigma^2 t_2+\sigma_{12}t_1+\sigma_{23}t_3+\sigma_{24}t_4\right].$$

Collecting terms gives

$$\frac{\partial^2 m}{\partial t_1 \partial t_2}=m(t_1,t_2,t_3,t_4)$$

$$\times\left[\sigma_{12}+\left(\mu+\sigma^2 t_1+\sigma_{12}t_2+\sigma_{13}t_3+\sigma_{14}t_4\right)\left(\mu+\sigma^2 t_2+\sigma_{12}t_1+\sigma_{23}t_3+\sigma_{24}t_4\right)\right], \quad \text{(B.4)}$$

yielding

$$\frac{\partial^2 m}{\partial t_1 \partial t_2}(0,0,0,0)=E[X_1X_2]=\sigma_{12}+\mu^2. \quad \text{(B.5)}$$

Also, by letting t_2 equal t_1 in equation (B.5), we get $E\left[X_1^2\right]=\sigma^2+\mu^2$.

The third partial derivative of m with respect to (t_1, t_2, t_3) is given by

$$\frac{\partial^3 m}{\partial t_1 \partial t_2 \partial t_3}=m(t_1,t_2,t_3,t_4)\left[\left(\mu+\sigma^2 t_1+\sigma_{12}t_2+\sigma_{13}t_3+\sigma_{14}t_4\right)\sigma_{23}\right.$$

$$\left.+\left(\mu+\sigma^2 t_2+\sigma_{12}t_1+\sigma_{23}t_3+\sigma_{24}t_4\right)\sigma_{13}\right]$$

$$+\left[\sigma_{12}+\left(\mu+\sigma^2 t_1+\sigma_{12}t_2+\sigma_{13}t_3+\sigma_{14}t_4\right)\left(\mu+\sigma^2 t_2+\sigma_{12}t_1+\sigma_{23}t_3+\sigma_{24}t_4\right)\right]$$

$$\times m(t_1,t_2,t_3,t_4)\left[\mu+\sigma^2 t_3+\sigma_{13}t_1+\sigma_{23}t_2+\sigma_{34}t_4\right].$$

Collecting terms gives

$$\frac{\partial^3 m}{\partial t_1 \partial t_2 \partial t_3}=m(t_1,t_2,t_3,t_4)\left[\sigma_{23}\left(\mu+\sigma^2 t_1+\sigma_{12}t_2+\sigma_{13}t_3+\sigma_{14}t_4\right)\right.$$

$$+\sigma_{13}\left(\mu+\sigma^2 t_2+\sigma_{12}t_1+\sigma_{23}t_3+\sigma_{24}t_4\right)+\sigma_{12}\left(\mu+\sigma^2 t_3+\sigma_{13}t_1+\sigma_{23}t_2+\sigma_{34}t_4\right)$$

$$+\left(\mu+\sigma^2 t_1+\sigma_{12}t_2+\sigma_{13}t_3+\sigma_{14}t_4\right)\left(\mu+\sigma^2 t_2+\sigma_{12}t_1+\sigma_{23}t_3+\sigma_{24}t_4\right)$$

$$\left.\times\left(\mu+\sigma^2 t_3+\sigma_{13}t_1+\sigma_{23}t_2+\sigma_{34}t_4\right)\right], \quad \text{(B.6)}$$

yielding

$$\frac{\partial^3 m}{\partial t_1 \partial t_2 \partial t_3}(0,0,0,0)=E[X_1X_2X_3]=\mu(\sigma_{12}+\sigma_{13}+\sigma_{23})+\mu^3. \quad \text{(B.7)}$$

By letting t_3 equal t_1 in equation (B.7), we get $E\left[X_1^2 X_2\right] = \mu\left(\sigma^2 + 2\sigma_{12}\right) + \mu^3$. Furthermore, by letting t_3 and t_2 each equal t_1 in equation (B.7), we get $E\left[X_1^3\right] = 3\mu\sigma^2 + \mu^3$.

The fourth partial derivative of m with respect to (t_1, t_2, t_3, t_4) is given by

$$\frac{\partial^4 m}{\partial t_1 \partial t_2 \partial t_3 \partial t_4} = m(t_1,t_2,t_3,t_4)[\sigma_{23}\sigma_{14} + \sigma_{13}\sigma_{24} + \sigma_{12}\sigma_{34}$$

$$+\sigma_{14}\left(\mu + \sigma^2 t_2 + \sigma_{12}t_1 + \sigma_{23}t_3 + \sigma_{24}t_4\right)\left(\mu + \sigma^2 t_3 + \sigma_{13}t_1 + \sigma_{23}t_2 + \sigma_{34}t_4\right)$$

$$+\sigma_{24}\left(\mu + \sigma^2 t_1 + \sigma_{12}t_2 + \sigma_{13}t_3 + \sigma_{14}t_4\right)\left(\mu + \sigma^2 t_3 + \sigma_{13}t_1 + \sigma_{23}t_2 + \sigma_{34}t_4\right)$$

$$+\sigma_{34}\left(\mu + \sigma^2 t_1 + \sigma_{12}t_2 + \sigma_{13}t_3 + \sigma_{14}t_4\right)\left(\mu + \sigma^2 t_2 + \sigma_{12}t_1 + \sigma_{23}t_3 + \sigma_{24}t_4\right)]$$

$$+m(t_1,t_2,t_3,t_4)\left[\mu + \sigma^2 t_4 + \sigma_{14}t_1 + \sigma_{24}t_2 + \sigma_{34}t_3\right]$$

$$\times[\sigma_{23}\left(\mu + \sigma^2 t_1 + \sigma_{12}t_2 + \sigma_{13}t_3 + \sigma_{14}t_4\right)$$

$$+\sigma_{13}\left(\mu + \sigma^2 t_2 + \sigma_{12}t_1 + \sigma_{23}t_3 + \sigma_{24}t_4\right) + \sigma_{12}\left(\mu + \sigma^2 t_3 + \sigma_{13}t_1 + \sigma_{23}t_2 + \sigma_{34}t_4\right)$$

$$+\left(\mu + \sigma^2 t_1 + \sigma_{12}t_2 + \sigma_{13}t_3 + \sigma_{14}t_4\right)\left(\mu + \sigma^2 t_2 + \sigma_{12}t_1 + \sigma_{23}t_3 + \sigma_{24}t_4\right)$$

$$\times\left(\mu + \sigma^2 t_3 + \sigma_{13}t_1 + \sigma_{23}t_2 + \sigma_{34}t_4\right)], \tag{B.8}$$

yielding

$$\frac{\partial^4 m}{\partial t_1 \partial t_2 \partial t_3 \partial t_4}(0,0,0,0) = E[X_1 X_2 X_3 X_4]$$

$$= \left[\sigma_{23}\sigma_{14} + \sigma_{13}\sigma_{24} + \sigma_{12}\sigma_{34} + \mu^2(\sigma_{14} + \sigma_{24} + \sigma_{34})\right]$$

$$+[\mu] \times \left[\mu(\sigma_{23} + \sigma_{13} + \sigma_{12}) + \mu^3\right].$$

Rearranging terms gives

$$\frac{\partial^4 m}{\partial t_1 \partial t_2 \partial t_3 \partial t_4}(0,0,0,0) = E[X_1 X_2 X_3 X_4]$$

$$= \sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{23}\sigma_{14} + \mu^2(\sigma_{12} + \sigma_{13} + \sigma_{14} + \sigma_{23} + \sigma_{24} + \sigma_{34}) + \mu^4. \tag{B.9}$$

By letting t_4 equal t_1 in equation (B.9), we get

$$E\left[X_1^2 X_2 X_3\right] = 2\sigma_{12}\sigma_{13} + \sigma_{23}\left(\sigma^2 + \mu^2\right) + 2\mu^2(\sigma_{12} + \sigma_{13}) + \mu^2\sigma^2 + \mu^4.$$

By letting t_4 and t_3 each equal t_1 in equation (B.9), we get

$$E\left[X_1^3 X_2\right] = 3\sigma_{12}\left(\sigma^2 + \mu^2\right) + 3\mu^2\sigma^2 + \mu^4.$$

By letting t_4 equal t_2 and letting t_3 equal t_1 in equation (B.9), we get

$$E\left[X_1^2 X_2^2\right] = \sigma^4 + 2\sigma_{12}^2 + 2\mu^2\left(\sigma^2 + 2\sigma_{12}\right) + \mu^4.$$

Finally, by letting t_4, t_3, and t_2 each equal t_1 in equation (B.9), we get

$$E\left[X_1^4\right] = 3\sigma^4 + 6\mu^2\sigma^2 + \mu^4.$$

APPENDIX C. THE AR(1) CORRELATION MATRIX

Suppose the stochastic process X_t obeys a stationary normal AR(1), that is, $(X_t - \mu) - \phi(X_{t-1} - \mu) = a_t$, where the a_t are *iid* $N\left(0, \sigma_a^2\right)$ and $-1 < \phi < 1$. It is well known that the variance of the process characteristic X_t is given by $Var[X_t] = \sigma^2 = \sigma_a^2 / \left(1 - \phi^2\right)$. Also, the covariance j periods apart is given by $Cov\left[X_t, X_{t-j}\right] = \phi^j Var[X_t] = \phi^j \sigma_a^2 / \left(1 - \phi^2\right)$ and the correlation j periods apart is given by $Corr\left[X_t, X_{t-j}\right] = \phi^j$.

Consider a sample $\{X_t\}_n$ from a stationary normal AR(1) process, consecutive in time, taken at the uniform time interval consistent with the parameter ϕ. The correlation matrix of the sample is given by

$$\begin{bmatrix} 1 & \phi & \phi^2 & \phi^3 & \cdots & \phi^{n-1} \\ \phi & 1 & \phi & \phi^2 & \cdots & \phi^{n-2} \\ \phi^2 & \phi & 1 & \phi & \cdots & \phi^{n-3} \\ \phi^3 & \phi^2 & \phi & 1 & \cdots & \phi^{n-4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi^{n-1} & \phi^{n-2} & \phi^{n-3} & \phi^{n-4} & \cdots & 1 \end{bmatrix}_{n \times n}$$

We are interested in the sum of the $n(n - 1)/2$ terms in the triangle below (or above) the main diagonal of ones. We denote this sum of terms as $\sum \textit{triangle}$.

Now

$$\begin{aligned} \sum \textit{triangle} &= (n-1)\phi + (n-2)\phi^2 + (n-3)\phi^3 + \cdots + \phi^{n-1} \\ &= (n\phi - \phi) + \left(n\phi^2 - 2\phi^2\right) + \left(n\phi^3 - 3\phi^3\right) + \cdots + \left(n\phi^{n-1} - (n-1)\phi^{n-1}\right) \\ &= \left(n\phi + n\phi^2 + n\phi^3 + \cdots + n\phi^{n-1}\right) - \left(\phi + 2\phi^2 + 3\phi^3 + \cdots + (n-1)\phi^{n-1}\right) \\ &= n\left(\phi + \phi^2 + \phi^3 + \cdots + \phi^{n-1}\right) - \left(\phi + 2\phi^2 + 3\phi^3 + \cdots + (n-1)\phi^{n-1}\right). \end{aligned} \tag{C.1}$$

The first group of terms on the right side of equation (C.1) can be rewritten as

$$n\left(\phi+\phi^2+\phi^3+\cdots+\phi^{n-1}\right)=n\left(1+\phi+\phi^2+\phi^3+\cdots+\phi^{n-1}-1\right)$$

$$=n\left\{\frac{1-\phi^n}{1-\phi}-1\right\}. \tag{C.2}$$

Now consider the second group of terms on the right side of equation (C.1),

$$\phi+2\phi^2+3\phi^3+\cdots+(n-1)\phi^{n-1}$$

$$=\phi\left(1+2\phi+3\phi^2+4\phi^3+\cdots+(n-1)\phi^{n-2}\right)$$

$$=\phi\left(0+1+2\phi+3\phi^2+4\phi^3+\cdots+(n-1)\phi^{n-2}\right)$$

$$=\phi\left\{\frac{d1}{d\phi}+\frac{d\phi}{d\phi}+\frac{d\phi^2}{d\phi}+\frac{d\phi^3}{d\phi}+\frac{d\phi^4}{d\phi}+\cdots+\frac{d\phi^{n-1}}{d\phi}\right\}$$

$$=\phi\left\{\frac{d}{d\phi}\left\{1+\phi+\phi^2+\phi^3+\phi^4+\cdots+\phi^{n-1}\right\}\right\}$$

$$=\phi\left\{\frac{d}{d\phi}\left\{\frac{1-\phi^n}{1-\phi}\right\}\right\}$$

$$=\phi\left\{\frac{\left(1-\phi^n\right)-n\phi^{n-1}(1-\phi)}{(1-\phi)^2}\right\}. \tag{C.3}$$

Substituting from equations (C.2) and (C.3) into equation (C.1) gives

$$\sum triangle=n\left\{\frac{1-\phi^n}{1-\phi}-1\right\}-\phi\left\{\frac{\left(1-\phi^n\right)-n\phi^{n-1}(1-\phi)}{(1-\phi)^2}\right\}$$

$$=n\left\{\frac{1-\phi^n}{1-\phi}-\frac{1-\phi}{1-\phi}\right\}-\phi\left\{\frac{\left(1-\phi^n\right)-n\phi^{n-1}(1-\phi)}{(1-\phi)^2}\right\}$$

$$=n\left\{\frac{1-\phi^n-1+\phi}{1-\phi}\right\}-\phi\left\{\frac{\left(1-\phi^n\right)-n\phi^{n-1}(1-\phi)}{(1-\phi)^2}\right\}$$

$$=n\left\{\frac{\phi-\phi^n}{1-\phi}\right\}-\phi\left\{\frac{\left(1-\phi^n\right)-n\phi^{n-1}(1-\phi)}{(1-\phi)^2}\right\}$$

$$= n(1-\phi)\left\{\frac{\phi-\phi^n}{(1-\phi)^2}\right\} - \phi\left\{\frac{(1-\phi^n) - n\phi^{n-1}(1-\phi)}{(1-\phi)^2}\right\}$$

$$= \frac{n(1-\phi)(\phi-\phi^n) - \phi(1-\phi^n) + n\phi^n(1-\phi)}{(1-\phi)^2}$$

$$= \frac{n(1-\phi)\phi - n(1-\phi)\phi^n - \phi(1-\phi^n) + n\phi^n(1-\phi)}{(1-\phi)^2}$$

$$= \frac{n\phi(1-\phi) - \phi(1-\phi^n)}{(1-\phi)^2}$$

$$= \frac{n\phi}{1-\phi} - \frac{\phi(1-\phi^n)}{(1-\phi)^2}$$

$$= \frac{\phi}{1-\phi}\left\{n - \frac{1-\phi^n}{1-\phi}\right\}. \tag{C.4}$$

Let $\bar{\rho} = \frac{2}{n(n-1)} \sum_{i<j}^{n(n-1)/2} \rho_{ij}$ denoted the average of these $n(n-1)/2$ terms. We have

$$\bar{\rho} = \frac{2}{n(n-1)} \sum_{i<j}^{n(n-1)/2} \rho_{ij} = \frac{2}{n(n-1)} \frac{\phi}{1-\phi}\left\{n - \frac{1-\phi^n}{1-\phi}\right\}. \tag{C.5}$$

Let $\bar{\rho}^2 = \frac{2}{n(n-1)} \sum_{i<j}^{n(n-1)/2} \rho_{ij}^2$ denote the average of the squares of these $n(n-1)/2$ terms. We then have, by substitution of ϕ^2 for ϕ into equation (C.5),

$$\bar{\rho}^2 = \frac{2}{n(n-1)} \sum_{i<j}^{n(n-1)/2} \rho_{ij}^2 = \frac{2}{n(n-1)} \frac{\phi^2}{1-\phi^2}\left\{n - \frac{1-\phi^{2n}}{1-\phi^2}\right\}. \tag{C.6}$$

We note that while stationarity considerations demand the restriction $-1 < \phi < 1$, equation (C.5) actually holds for any real ϕ not equal to 1, while equation (C.6) holds for any real ϕ not equal to 1 or -1. Of course, positive integer n is at least 2.

APPENDIX D. DERIVATION OF AN IMPORTANT VARIANCE

Let $\{X_i\}_n$ be multivariate normal with equal means μ, equal variances σ^2, and covariances $\sigma(i, j)$ not necessarily zero. We seek $Var\left[\sum_{i=1}^{n}(X_i - \bar{X})^2\right]$.

Now

$$Var\left[\sum_{i=1}^{n}(X_i - \bar{X})^2\right] = Var\left[\sum_{i=1}^{n} X_i^2 - \frac{1}{n}\sum_{i=1}^{n} X_i \sum_{i=1}^{n} X_i\right].$$

Using the identity $Var[Z] = E\left[Z^2\right] - E^2[Z]$, let $Z = \sum_{i=1}^{n} X_i^2 - \frac{1}{n}\sum_{i=1}^{n} X_i \sum_{i=1}^{n} X_i$, giving

$$Var\left[\sum_{i=1}^{n} X_i^2 - \frac{1}{n}\sum_{i=1}^{n} X_i \sum_{i=1}^{n} X_i\right] = E\left[\left\{\sum_{i=1}^{n} X_i^2 - \frac{1}{n}\sum_{i=1}^{n} X_i \sum_{i=1}^{n} X_i\right\}^2\right]$$

$$-E^2\left[\sum_{i=1}^{n} X_i^2 - \frac{1}{n}\sum_{i=1}^{n} X_i \sum_{i=1}^{n} X_i\right]. \qquad \text{(D.1)}$$

We have previously derived the second expectation as

$$E\left[\sum_{i=1}^{n} X_i^2 - \frac{1}{n}\sum_{i=1}^{n} X_i \sum_{i=1}^{n} X_i\right] = (n-1)(1-\bar{\rho})\sigma^2.$$

and so

$$E^2\left[\sum_{i=1}^{n} X_i^2 - \frac{1}{n}\sum_{i=1}^{n} X_i \sum_{i=1}^{n} X_i\right] = \left\{(n-1)(1-\bar{\rho})\sigma^2\right\}^2 = (n-1)^2(1-\bar{\rho})^2\sigma^4. \qquad \text{(D.2)}$$

We rewrite the first expectation on the right side of equation (D.1) as

$$E\left[\left\{\sum_{i=1}^{n} X_i^2 - \frac{1}{n}\sum_{i=1}^{n} X_i \sum_{i=1}^{n} X_i\right\}^2\right] = E\left[\sum_{i=1}^{n} X_i^2 \sum_{i=1}^{n} X_i^2\right]$$

$$-\frac{2}{n}E\left[\sum_{i=1}^{n} X_i^2 \sum_{i=1}^{n} X_i \sum_{i=1}^{n} X_i\right] + \frac{1}{n^2}E\left[\sum_{i=1}^{n} X_i \sum_{i=1}^{n} X_i \sum_{i=1}^{n} X_i \sum_{i=1}^{n} X_i\right]. \qquad \text{(D.3)}$$

Substituting $\sum_{i=1}^{n} X_i^2 \sum_{i=1}^{n} X_i \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} X_i^2 \left\{ \sum_{i=1}^{n} X_i^2 + \sum_{i \neq j}^{n(n-1)} X_i X_j \right\}$ into equation (D.3) gives

$$E\left[\left\{\sum_{i=1}^{n} X_i^2 - \frac{1}{n}\sum_{i=1}^{n} X_i \sum_{i=1}^{n} X_i\right\}^2\right] = E\left[\sum_{i=1}^{n} X_i^2 \sum_{i=1}^{n} X_i^2\right]$$

$$-\frac{2}{n}E\left[\sum_{i=1}^{n} X_i^2 \left\{\sum_{i=1}^{n} X_i^2 + \sum_{i \neq j}^{n(n-1)} X_i X_j\right\}\right] + \frac{1}{n^2}E\left[\sum_{i=1}^{n} X_i \sum_{i=1}^{n} X_i \sum_{i=1}^{n} X_i \sum_{i=1}^{n} X_i\right]. \tag{D.4}$$

Expanding equation (D.4) gives

$$E\left[\left\{\sum_{i=1}^{n} X_i^2 - \frac{1}{n}\sum_{i=1}^{n} X_i \sum_{i=1}^{n} X_i\right\}^2\right] = E\left[\sum_{i=1}^{n} X_i^2 \sum_{i=1}^{n} X_i^2\right] - \frac{2}{n}E\left[\sum_{i=1}^{n} X_i^2 \sum_{i=1}^{n} X_i^2\right]$$

$$-\frac{2}{n}E\left[\sum_{i=1}^{n} X_i^2 \sum_{i \neq j}^{n(n-1)} X_i X_j\right] + \frac{1}{n^2}E\left[\sum_{i=1}^{n} X_i \sum_{i=1}^{n} X_i \sum_{i=1}^{n} X_i \sum_{i=1}^{n} X_i\right]. \tag{D.5}$$

Substituting

$$\sum_{i=1}^{n} X_i \sum_{i=1}^{n} X_i \sum_{i=1}^{n} X_i \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} X_i^2 \sum_{i=1}^{n} X_i^2 + 2\sum_{i=1}^{n} X_i^2 \sum_{i \neq j}^{n(n-1)} X_i X_j + \sum_{i \neq j}^{n(n-1)} X_i X_j \sum_{i \neq j}^{n(n-1)} X_i X_j$$

into equation (D.5) and collecting terms gives

$$E\left[\left\{\sum_{i=1}^{n} X_i^2 - \frac{1}{n}\sum_{i=1}^{n} X_i \sum_{i=1}^{n} X_i\right\}^2\right] = \left\{1 - \frac{2}{n}\right\}E\left[\sum_{i=1}^{n} X_i^2 \sum_{i=1}^{n} X_i^2\right] - \frac{2}{n}E\left[\sum_{i=1}^{n} X_i^2 \sum_{i \neq j}^{n(n-1)} X_i X_j\right]$$

$$+\frac{1}{n^2}E\left[\sum_{i=1}^{n} X_i^2 \sum_{i=1}^{n} X_i^2 + 2\sum_{i=1}^{n} X_i^2 \sum_{i \neq j}^{n(n-1)} X_i X_j + \sum_{i \neq j}^{n(n-1)} X_i X_j \sum_{i \neq j}^{n(n-1)} X_i X_j\right]. \tag{D.6}$$

Distributing the last terms of equation (D.6) gives

$$E\left[\left\{\sum_{i=1}^{n} X_i^2 - \frac{1}{n}\sum_{i=1}^{n} X_i \sum_{i=1}^{n} X_i\right\}^2\right] = \left\{1 - \frac{2}{n}\right\}E\left[\sum_{i=1}^{n} X_i^2 \sum_{i=1}^{n} X_i^2\right] - \frac{2}{n}E\left[\sum_{i=1}^{n} X_i^2 \sum_{i \neq j}^{n(n-1)} X_i X_j\right]$$

$$+\frac{1}{n^2}\left\{E\left[\sum_{i=1}^{n} X_i^2 \sum_{i=1}^{n} X_i^2\right] + 2E\left[\sum_{i=1}^{n} X_i^2 \sum_{i \neq j}^{n(n-1)} X_i X_j\right] + E\left[\sum_{i \neq j}^{n(n-1)} X_i X_j \sum_{i \neq j}^{n(n-1)} X_i X_j\right]\right\}. \tag{D.7}$$

Substituting

$$\sum_{i\neq j}^{n(n-1)} X_i X_j \sum_{i\neq j}^{n(n-1)} X_i X_j = \sum_{i\neq j}^{2n(n-1)} X_i^2 X_j^2 + \sum_{i\neq j\neq k}^{4n(n-1)(n-2)} X_i^2 X_j X_k + \sum_{i\neq j\neq k\neq l}^{n(n-1)(n-2)(n-3)} X_i X_j X_k X_l$$

into equation (D.7) gives

$$E\left[\left\{\sum_{i=1}^{n} X_i^2 - \frac{1}{n}\sum_{i=1}^{n} X_i \sum_{i=1}^{n} X_i\right\}^2\right] = \left\{1-\frac{2}{n}\right\} E\left[\sum_{i=1}^{n} X_i^2 \sum_{i=1}^{n} X_i^2\right] - \frac{2}{n} E\left[\sum_{i=1}^{n} X_i^2 \sum_{i\neq j}^{n(n-1)} X_i X_j\right]$$
$$+\frac{1}{n^2}\left\{E\left[\sum_{i=1}^{n} X_i^2 \sum_{i=1}^{n} X_i^2\right] + 2E\left[\sum_{i=1}^{n} X_i^2 \sum_{i\neq j}^{n(n-1)} X_i X_j\right]\right\}$$
$$+\frac{1}{n^2} E\left[\sum_{i\neq j}^{2n(n-1)} X_i^2 X_j^2 + \sum_{i\neq j\neq k}^{4n(n-1)(n-2)} X_i^2 X_j X_k + \sum_{i\neq j\neq k\neq l}^{n(n-1)(n-2)(n-3)} X_i X_j X_k X_l\right]. \qquad \text{(D.8)}$$

Distributing the last line of expectations of equation (D.8) gives

$$E\left[\left\{\sum_{i=1}^{n} X_i^2 - \frac{1}{n}\sum_{i=1}^{n} X_i \sum_{i=1}^{n} X_i\right\}^2\right] = \left\{1-\frac{2}{n}\right\} E\left[\sum_{i=1}^{n} X_i^2 \sum_{i=1}^{n} X_i^2\right] - \frac{2}{n} E\left[\sum_{i=1}^{n} X_i^2 \sum_{i\neq j}^{n(n-1)} X_i X_j\right]$$
$$+\frac{1}{n^2}\left\{E\left[\sum_{i=1}^{n} X_i^2 \sum_{i=1}^{n} X_i^2\right] + 2E\left[\sum_{i=1}^{n} X_i^2 \sum_{i\neq j}^{n(n-1)} X_i X_j\right]\right\} + \frac{1}{n^2} E\left[\sum_{i\neq j}^{2n(n-1)} X_i^2 X_j^2\right]$$
$$+\frac{1}{n^2} E\left[\sum_{i\neq j\neq k}^{4n(n-1)(n-2)} X_i^2 X_j X_k\right] + \frac{1}{n^2} E\left[\sum_{i\neq j\neq k\neq l}^{n(n-1)(n-2)(n-3)} X_i X_j X_k X_l\right]. \qquad \text{(D.9)}$$

Collecting the terms of equation (D.9) gives

$$E\left[\left\{\sum_{i=1}^{n} X_i^2 - \frac{1}{n}\sum_{i=1}^{n} X_i \sum_{i=1}^{n} X_i\right\}^2\right] = \left\{1-\frac{2}{n}+\frac{1}{n^2}\right\} E\left[\sum_{i=1}^{n} X_i^2 \sum_{i=1}^{n} X_i^2\right]$$
$$-\left\{\frac{2}{n}-\frac{2}{n^2}\right\} E\left[\sum_{i=1}^{n} X_i^2 \sum_{i\neq j}^{n(n-1)} X_i X_j\right] + \frac{1}{n^2} E\left[\sum_{i\neq j}^{2n(n-1)} X_i^2 X_j^2\right]$$
$$+\frac{1}{n^2} E\left[\sum_{i\neq j\neq k}^{4n(n-1)(n-2)} X_i^2 X_j X_k\right] + \frac{1}{n^2} E\left[\sum_{i\neq j\neq k\neq l}^{n(n-1)(n-2)(n-3)} X_i X_j X_k X_l\right]. \qquad \text{(D.10)}$$

Substituting

$$\sum_{i=1}^{n} X_i^2 \sum_{i=1}^{n} X_i^2 = \sum_{i=1}^{n} X_i^4 + \sum_{i \neq j}^{n(n-1)} X_i^2 X_j^2$$

and

$$\sum_{i=1}^{n} X_i^2 \sum_{i \neq j}^{n(n-1)} X_i X_j = \sum_{i \neq j}^{2n(n-1)} X_i^3 X_j + \sum_{i \neq j \neq k}^{n(n-1)(n-2)} X_i^2 X_j X_k$$

into equation (D.10) gives

$$E\left[\left\{\sum_{i=1}^{n} X_i^2 - \frac{1}{n}\sum_{i=1}^{n} X_i \sum_{i=1}^{n} X_i\right\}^2\right] = \left\{1 - \frac{1}{n}\right\}^2 E\left[\sum_{i=1}^{n} X_i^4 + \sum_{i \neq j}^{n(n-1)} X_i^2 X_j^2\right]$$
$$-\frac{2}{n}\left\{1-\frac{1}{n}\right\} E\left[\sum_{i \neq j}^{2n(n-1)} X_i^3 X_j + \sum_{i \neq j \neq k}^{n(n-1)(n-2)} X_i^2 X_j X_k\right] + \frac{1}{n^2} E\left[\sum_{i \neq j}^{2n(n-1)} X_i^2 X_j^2\right]$$
$$+\frac{1}{n^2} E\left[\sum_{i \neq j \neq k}^{4n(n-1)(n-2)} X_i^2 X_j X_k\right] + \frac{1}{n^2} E\left[\sum_{i \neq j \neq k \neq l}^{n(n-1)(n-2)(n-3)} X_i X_j X_k X_l\right]. \qquad \text{(D.11)}$$

Collecting the terms of equation (D.11) gives

$$E\left[\left\{\sum_{i=1}^{n} X_i^2 - \frac{1}{n}\sum_{i=1}^{n} X_i \sum_{i=1}^{n} X_i\right\}^2\right] = \left\{1 - \frac{1}{n}\right\}^2 E\left[\sum_{i=1}^{n} X_i^4\right]$$
$$+\left\{\left\{1-\frac{1}{n}\right\}^2 + \frac{2}{n^2}\right\} E\left[\sum_{i \neq j}^{n(n-1)} X_i^2 X_j^2\right] - \frac{4}{n}\left\{1-\frac{1}{n}\right\} E\left[\sum_{i \neq j}^{n(n-1)} X_i^3 X_j\right]$$
$$+\left\{\frac{4}{n^2} - \frac{2}{n}\left\{1-\frac{1}{n}\right\}\right\} E\left[\sum_{i \neq j \neq k}^{n(n-1)(n-2)} X_i^2 X_j X_k\right] + \frac{1}{n^2} E\left[\sum_{i \neq j \neq k \neq l}^{n(n-1)(n-2)(n-3)} X_i X_j X_k X_l\right]. \qquad \text{(D.12)}$$

Distributing expectations across the sums in equation (D.12) gives

$$E\left[\left\{\sum_{i=1}^{n} X_i^2 - \frac{1}{n}\sum_{i=1}^{n} X_i \sum_{i=1}^{n} X_i\right\}^2\right] = \left\{1 - \frac{1}{n}\right\}^2 \sum_{i=1}^{n} E\left[X_i^4\right]$$

$$+\left\{\left\{1-\frac{1}{n}\right\}^2+\frac{2}{n^2}\right\}\sum_{i\neq j}^{n(n-1)} E\left[X_i^2 X_j^2\right]-\frac{4}{n}\left\{1-\frac{1}{n}\right\}\sum_{i\neq j}^{n(n-1)} E\left[X_i^3 X_j\right]$$

$$+\left\{\frac{4}{n^2}-\frac{2}{n}\left\{1-\frac{1}{n}\right\}\right\}\sum_{i\neq j\neq k}^{n(n-1)(n-2)} E\left[X_i^2 X_j X_k\right]$$

$$+\frac{1}{n^2}\sum_{i\neq j\neq k\neq l}^{n(n-1)(n-2)(n-3)} E\left[X_i X_j X_k X_l\right]. \tag{D.13}$$

We are now ready to substitute moments into equation (D.13). From Appendix B,

$$E\left[X_i^4\right]=3\sigma^4+6\mu^2\sigma^2+\mu^4$$

$$E\left[X_i^3 X_j\right]=3\sigma^2\rho_{ij}\left(\sigma^2+\mu^2\right)+3\mu^2\sigma^2+\mu^4$$

$$E\left[X_i^2 X_j^2\right]=\sigma^4\left(1+2\rho_{ij}^2\right)+2\mu^2\sigma^2\left(1+2\rho_{ij}\right)+\mu^4$$

$$E\left[X_i^2 X_j X_k\right]=2\sigma^4\rho_{ij}\rho_{ik}+\sigma^2\rho_{jk}\left(\sigma^2+\mu^2\right)+2\mu^2\sigma^2\left(\rho_{ij}+\rho_{ik}\right)+\mu^2\sigma^2+\mu^4$$

$$E\left[X_i X_j X_k X_l\right]=\sigma^4\left(\rho_{ij}\rho_{kl}+\rho_{ik}\rho_{jl}+\rho_{jk}\rho_{il}\right)$$

$$+\mu^2\sigma^2\left(\rho_{ij}+\rho_{ik}+\rho_{il}+\rho_{jk}+\rho_{jl}+\rho_{kl}\right)+\mu^4.$$

Substituting into equation (D.13) gives

$$E\left[\left\{\sum_{i=1}^{n} X_i^2-\frac{1}{n}\sum_{i=1}^{n} X_i\sum_{i=1}^{n} X_i\right\}^2\right]=\left\{1-\frac{1}{n}\right\}^2\sum_{i=1}^{n}\left\{3\sigma^4+6\mu^2\sigma^2+\mu^4\right\}$$

$$+\left\{\left\{1-\frac{1}{n}\right\}^2+\frac{2}{n^2}\right\}\sum_{i\neq j}^{n(n-1)}\left\{\sigma^4\left(1+2\rho_{ij}^2\right)+2\mu^2\sigma^2\left(1+2\rho_{ij}\right)+\mu^4\right\}$$

$$-\frac{4}{n}\left\{1-\frac{1}{n}\right\}\sum_{i\neq j}^{n(n-1)}\left\{3\sigma^2\rho_{ij}\left(\sigma^2+\mu^2\right)+3\mu^2\sigma^2+\mu^4\right\}$$

$$+\left\{\frac{4}{n^2}-\frac{2}{n}\left\{1-\frac{1}{n}\right\}\right\}\sum_{i\neq j\neq k}^{n(n-1)(n-2)}\left\{\begin{array}{l}2\sigma^4\rho_{ij}\rho_{ik}+\sigma^2\rho_{jk}\left(\sigma^2+\mu^2\right)\\+2\mu^2\sigma^2\left(\rho_{ij}+\rho_{ik}\right)+\mu^2\sigma^2+\mu^4\end{array}\right\}$$

$$+\frac{1}{n^2}\sum_{i\neq j\neq k\neq l}^{n(n-1)(n-2)(n-3)}\left\{\begin{array}{l}\sigma^4\left(\rho_{ij}\rho_{kl}+\rho_{ik}\rho_{jl}+\rho_{jk}\rho_{il}\right)\\+\mu^2\sigma^2\left(\rho_{ij}+\rho_{ik}+\rho_{il}+\rho_{jk}+\rho_{jl}+\rho_{kl}\right)+\mu^4\end{array}\right\}. \tag{D.14}$$

For later computational clarity, we rewrite the lead coefficients of equation (D.14) as

$$E\left[\left\{\sum_{i=1}^{n} X_i^2 - \frac{1}{n}\sum_{i=1}^{n} X_i \sum_{i=1}^{n} X_i\right\}^2\right] = \left\{1-\frac{2}{n}+\frac{1}{n^2}\right\}\sum_{i=1}^{n}\left\{3\sigma^4 + 6\mu^2\sigma^2 + \mu^4\right\}$$

$$+\left\{1-\frac{2}{n}+\frac{3}{n^2}\right\}\sum_{i\neq j}^{n(n-1)}\left\{\sigma^4\left(1+2\rho_{ij}^2\right)+2\mu^2\sigma^2\left(1+2\rho_{ij}\right)+\mu^4\right\}$$

$$+\left\{\frac{4}{n^2}-\frac{4}{n}\right\}\sum_{i\neq j}^{n(n-1)}\left\{3\sigma^2\rho_{ij}\left(\sigma^2+\mu^2\right)+3\mu^2\sigma^2+\mu^4\right\}$$

$$+\left\{\frac{6}{n^2}-\frac{2}{n}\right\}\sum_{i\neq j\neq k}^{n(n-1)(n-2)}\left\{\begin{array}{l}2\sigma^4\rho_{ij}\rho_{ik}+\sigma^2\rho_{jk}\left(\sigma^2+\mu^2\right)\\ +2\mu^2\sigma^2\left(\rho_{ij}+\rho_{ik}\right)+\mu^2\sigma^2+\mu^4\end{array}\right\}$$

$$+\frac{1}{n^2}\sum_{i\neq j\neq k\neq l}^{n(n-1)(n-2)(n-3)}\left\{\begin{array}{l}\sigma^4\left(\rho_{ij}\rho_{kl}+\rho_{ik}\rho_{jl}+\rho_{jk}\rho_{il}\right)\\ +\mu^2\sigma^2\left(\rho_{ij}+\rho_{ik}+\rho_{il}+\rho_{jk}+\rho_{jl}+\rho_{kl}\right)+\mu^4\end{array}\right\}. \qquad \text{(D.15)}$$

To begin the simplification of equation (D.15), we examine the coefficients of the terms involving only μ^4. They are

$$\left\{1-\frac{2}{n}+\frac{1}{n^2}\right\}n+\left\{1-\frac{2}{n}+\frac{3}{n^2}\right\}n(n-1)+\left\{\frac{4}{n^2}-\frac{4}{n}\right\}n(n-1)$$
$$+\left\{\frac{6}{n^2}-\frac{2}{n}\right\}n(n-1)(n-2)+\frac{1}{n^2}n(n-1)(n-2)(n-3)$$

is equal to

$$\left\{1-\frac{2}{n}+\frac{1}{n^2}\right\}n+\left\{1-\frac{6}{n}+\frac{7}{n^2}\right\}n(n-1)$$
$$+\left\{\frac{6}{n^2}-\frac{2}{n}\right\}n(n-1)(n-2)+\frac{1}{n^2}n(n-1)(n-2)(n-3)$$

is equal to

$$\left\{n-2+\frac{1}{n}\right\}+\left\{1-\frac{6}{n}+\frac{7}{n^2}\right\}\left(n^2-n\right)$$
$$+\left\{\frac{6}{n^2}-\frac{2}{n}\right\}\left(n^3-3n^2+2n\right)+\frac{1}{n^2}\left(n^4-6n^3+11n^2-6n\right)$$

is equal to

$$\left\{n-2+\frac{1}{n}\right\}+\left\{n^2-7n+13-\frac{7}{n}\right\}$$
$$+\left\{-2n^2+12n-22+\frac{12}{n}\right\}+\left\{n^2-6n+11-\frac{6}{n}\right\}$$

is equal to

$$n^2(1-2+1)+n(1-7+12-6)+1(-2+13-22+11)+\frac{1}{n}(1-7+12-6)$$
$$=n^2(0)+n(0)+1(0)+\frac{1}{n}(0)=0+0+0+0=0.$$

We see that the μ^4 terms drop out, as they should.

Returning to equation (D.15), we next examine the coefficients of the terms involving only $\mu^2\sigma^2$. They are

$$\left\{1-\frac{2}{n}+\frac{1}{n^2}\right\}6n+\left\{1-\frac{2}{n}+\frac{3}{n^2}\right\}2n(n-1)$$
$$+\left\{\frac{4}{n^2}-\frac{4}{n}\right\}3n(n-1)+\left\{\frac{6}{n^2}-\frac{2}{n}\right\}n(n-1)(n-2)$$

is equal to

$$\left\{6-\frac{12}{n}+\frac{6}{n^2}\right\}n+\left\{2-\frac{16}{n}+\frac{18}{n^2}\right\}n(n-1)+\left\{\frac{6}{n^2}-\frac{2}{n}\right\}n(n-1)(n-2)$$

is equal to

$$\left\{6-\frac{12}{n}+\frac{6}{n^2}\right\}n+\left\{2-\frac{16}{n}+\frac{18}{n^2}\right\}\left(n^2-n\right)+\left\{\frac{6}{n^2}-\frac{2}{n}\right\}\left(n^3-3n^2+2n\right)$$

is equal to

$$\left\{6n-12+\frac{6}{n}\right\}+\left\{2n^2-18n+34-\frac{18}{n}\right\}+\left\{-2n^2+12n-22+\frac{12}{n}\right\}$$

is equal to

$$n^2(2-2)+n(6-18+12)+1(-12+34-22)+\frac{1}{n}(6-18+12)$$

$$= n^2(0) + n(0) + 1(0) + \frac{1}{n}(0) = 0 + 0 + 0 + 0 = 0.$$

We see that the $\mu^2\sigma^2$ terms drop out, as they should.

Returning to equation (D.15), we next examine the terms involving only $\mu^2\sigma^2\rho_{ij}$. They are

$$\left\{1 - \frac{2}{n} + \frac{3}{n^2}\right\}\sum_{i \neq j}^{n(n-1)} 2\mu^2\sigma^2 2\rho_{ij} + \left\{\frac{4}{n^2} - \frac{4}{n}\right\}\sum_{i \neq j}^{n(n-1)} 3\sigma^2\rho_{ij}\mu^2$$

$$+\left\{\frac{6}{n^2} - \frac{2}{n}\right\}\sum_{i \neq j \neq k}^{n(n-1)(n-2)} \left\{\mu^2\sigma^2\rho_{jk} + 2\mu^2\sigma^2\left(\rho_{ij} + \rho_{ik}\right)\right\}$$

$$+\frac{1}{n^2}\sum_{i \neq j \neq k \neq l}^{n(n-1)(n-2)(n-3)} \left\{\mu^2\sigma^2\left(\rho_{ij} + \rho_{ik} + \rho_{il} + \rho_{jk} + \rho_{jl} + \rho_{kl}\right)\right\}$$

is equal to

$$\left\{1 - \frac{2}{n} + \frac{3}{n^2}\right\}4\mu^2\sigma^2\sum_{i \neq j}^{n(n-1)}\rho_{ij} + \left\{\frac{4}{n^2} - \frac{4}{n}\right\}3\mu^2\sigma^2\sum_{i \neq j}^{n(n-1)}\rho_{ij}$$

$$+\left\{\frac{6}{n^2} - \frac{2}{n}\right\}\mu^2\sigma^2\sum_{i \neq j \neq k}^{n(n-1)(n-2)}\rho_{jk} + \left\{\frac{6}{n^2} - \frac{2}{n}\right\}2\mu^2\sigma^2\sum_{i \neq j \neq k}^{n(n-1)(n-2)}\rho_{ij}$$

$$+\left\{\frac{6}{n^2} - \frac{2}{n}\right\}2\mu^2\sigma^2\sum_{i \neq j \neq k}^{n(n-1)(n-2)}\rho_{ik}$$

$$+\frac{1}{n^2}\mu^2\sigma^2\sum_{i \neq j \neq k \neq l}^{n(n-1)(n-2)(n-3)}\rho_{ij} + \frac{1}{n^2}\mu^2\sigma^2\sum_{i \neq j \neq k \neq l}^{n(n-1)(n-2)(n-3)}\rho_{ik}$$

$$+\frac{1}{n^2}\mu^2\sigma^2\sum_{i \neq j \neq k \neq l}^{n(n-1)(n-2)(n-3)}\rho_{il} + \frac{1}{n^2}\mu^2\sigma^2\sum_{i \neq j \neq k \neq l}^{n(n-1)(n-2)(n-3)}\rho_{jk}$$

$$+\frac{1}{n^2}\mu^2\sigma^2\sum_{i \neq j \neq k \neq l}^{n(n-1)(n-2)(n-3)}\rho_{jl} + \frac{1}{n^2}\mu^2\sigma^2\sum_{i \neq j \neq k \neq l}^{n(n-1)(n-2)(n-3)}\rho_{kl}$$

is equal to

$$\left\{1 - \frac{2}{n} + \frac{3}{n^2}\right\}4\mu^2\sigma^2\sum_{i \neq j}^{n(n-1)}\rho_{ij} + \left\{\frac{4}{n^2} - \frac{4}{n}\right\}3\mu^2\sigma^2\sum_{i \neq j}^{n(n-1)}\rho_{ij}$$

$$+\left\{\frac{6}{n^2}-\frac{2}{n}\right\}(n-2)\mu^2\sigma^2\sum_{i\neq j}^{n(n-1)}\rho_{ij}+\left\{\frac{6}{n^2}-\frac{2}{n}\right\}2(n-2)\mu^2\sigma^2\sum_{i\neq j}^{n(n-1)}\rho_{ij}$$

$$+\left\{\frac{6}{n^2}-\frac{2}{n}\right\}2(n-2)\mu^2\sigma^2\sum_{i\neq j}^{n(n-1)}\rho_{ij}$$

$$+\frac{1}{n^2}(n-2)(n-3)\mu^2\sigma^2\sum_{i\neq j}^{n(n-1)}\rho_{ij}+\frac{1}{n^2}(n-2)(n-3)\mu^2\sigma^2\sum_{i\neq j}^{n(n-1)}\rho_{ij}$$

$$+\frac{1}{n^2}(n-2)(n-3)\mu^2\sigma^2\sum_{i\neq j}^{n(n-1)}\rho_{ij}+\frac{1}{n^2}(n-2)(n-3)\mu^2\sigma^2\sum_{i\neq j}^{n(n-1)}\rho_{ij}$$

$$+\frac{1}{n^2}(n-2)(n-3)\mu^2\sigma^2\sum_{i\neq j}^{n(n-1)}\rho_{ij}+\frac{1}{n^2}(n-2)(n-3)\mu^2\sigma^2\sum_{i\neq j}^{n(n-1)}\rho_{ij}$$

is equal to

$$\left\{4-\frac{8}{n}+\frac{12}{n^2}\right\}\mu^2\sigma^2\sum_{i\neq j}^{n(n-1)}\rho_{ij}+\left\{\frac{12}{n^2}-\frac{12}{n}\right\}\mu^2\sigma^2\sum_{i\neq j}^{n(n-1)}\rho_{ij}$$

$$+\left\{\frac{30}{n^2}-\frac{10}{n}\right\}(n-2)\mu^2\sigma^2\sum_{i\neq j}^{n(n-1)}\rho_{ij}+\frac{6}{n^2}(n-2)(n-3)\mu^2\sigma^2\sum_{i\neq j}^{n(n-1)}\rho_{ij}.$$

Collecting the coefficients of $\mu^2\sigma^2\sum_{i\neq j}^{n(n-1)}\rho_{ij}$, we have

$$\left\{4-\frac{8}{n}+\frac{12}{n^2}\right\}+\left\{\frac{12}{n^2}-\frac{12}{n}\right\}+\left\{\frac{30}{n^2}-\frac{10}{n}\right\}(n-2)+\frac{6}{n^2}(n-2)(n-3)$$

is equal to

$$\left\{4-\frac{8}{n}+\frac{12}{n^2}\right\}+\left\{\frac{12}{n^2}-\frac{12}{n}\right\}+\left\{\frac{50}{n}-\frac{60}{n^2}-10\right\}+\left\{6-\frac{30}{n}+\frac{36}{n^2}\right\}$$

is equal to

$$1(4-10+6)+\frac{1}{n}(-8-12+50-30)+\frac{1}{n^2}(12+12-60+36)$$

$$=1(0)+\frac{1}{n}(0)+\frac{1}{n^2}(0)=0+0+0=0.$$

We see that the $\mu^2\sigma^2\rho_{ij}$ terms drop out, as they should.

Returning to equation (D.15), we next examine the terms involving only σ^4. They are

$$\left\{1-\frac{2}{n}+\frac{1}{n^2}\right\}\sum_{i=1}^{n}3\sigma^4+\left\{1-\frac{2}{n}+\frac{3}{n^2}\right\}\sum_{i\neq j}^{n(n-1)}\sigma^4$$

$$=\left\{1-\frac{2}{n}+\frac{1}{n^2}\right\}3n\sigma^4+\left\{1-\frac{2}{n}+\frac{3}{n^2}\right\}n(n-1)\sigma^4$$

$$=\left\{1-\frac{2}{n}+\frac{1}{n^2}\right\}3n\sigma^4+\left\{1-\frac{2}{n}+\frac{3}{n^2}\right\}\left(n^2-n\right)\sigma^4$$

$$=\left\{3n-6+\frac{3}{n}\right\}\sigma^4+\left\{n^2-3n+5-\frac{3}{n}\right\}\sigma^4$$

$$=\left(n^2-1\right)\sigma^4. \tag{D.16}$$

Next, consider the $\sigma^4\rho_{ij}$ terms in equation (D.15). They are

$$\left\{\frac{4}{n^2}-\frac{4}{n}\right\}\sum_{i\neq j}^{n(n-1)}3\sigma^4\rho_{ij}+\left\{\frac{6}{n^2}-\frac{2}{n}\right\}\sum_{i\neq j\neq k}^{n(n-1)(n-2)}\sigma^4\rho_{ij}$$

$$=\left\{\frac{12}{n^2}-\frac{12}{n}\right\}\sigma^4\sum_{i\neq j}^{n(n-1)}\rho_{ij}+\left\{\frac{6}{n^2}-\frac{2}{n}\right\}\sigma^4\sum_{i\neq j\neq k}^{n(n-1)(n-2)}\rho_{jk}$$

$$=\left\{\frac{12}{n^2}-\frac{12}{n}\right\}\sigma^4\sum_{i\neq j}^{n(n-1)}\rho_{ij}+\left\{\frac{6}{n^2}-\frac{2}{n}\right\}(n-2)\sigma^4\sum_{i\neq j}^{n(n-1)}\rho_{ij}$$

$$=\left\{\frac{12}{n^2}-\frac{12}{n}\right\}\sigma^4\sum_{i\neq j}^{n(n-1)}\rho_{ij}+\left\{-\frac{12}{n^2}+\frac{10}{n}-2\right\}\sigma^4\sum_{i\neq j}^{n(n-1)}\rho_{ij}$$

$$=\left\{-\frac{2}{n}-2\right\}\sigma^4\sum_{i\neq j}^{n(n-1)}\rho_{ij}. \tag{D.17}$$

Next, consider the $\sigma^4\rho_{ij}^2$ terms in equation (D.15). They are

$$\left\{1-\frac{2}{n}+\frac{3}{n^2}\right\}\sum_{i\neq j}^{n(n-1)}2\sigma^4\rho_{ij}^2=\left\{2-\frac{4}{n}+\frac{6}{n^2}\right\}\sigma^4\sum_{i\neq j}^{n(n-1)}\rho_{ij}^2. \tag{D.18}$$

Next, consider the $\sigma^4\rho_{ij}\rho_{ik}$ terms in equation (D.15). They are

$$\left\{\frac{6}{n^2}-\frac{2}{n}\right\}\sum_{i\neq j\neq k}^{n(n-1)(n-2)}2\sigma^4\rho_{ij}\rho_{ik}=\left\{\frac{12}{n^2}-\frac{4}{n}\right\}\sigma^4\sum_{i\neq j\neq k}^{n(n-1)(n-2)}\rho_{ij}\rho_{ik}. \tag{D.19}$$

Next, consider the $\sigma^4\rho_{ij}\rho_{kl}$ terms in equation (D.15). They are

$$\frac{1}{n^2}\sum_{i\neq j\neq k\neq l}^{n(n-1)(n-2)(n-3)}\sigma^4\left(\rho_{ij}\rho_{kl}+\rho_{ik}\rho_{jl}+\rho_{jk}\rho_{il}\right)=\frac{3}{n^2}\sigma^4\sum_{i\neq j\neq k\neq l}^{n(n-1)(n-2)(n-3)}\rho_{ij}\rho_{kl}. \tag{D.20}$$

Recall that

$$E^2\left[\sum_{i=1}^{n}X_i^2-\frac{1}{n}\sum_{i=1}^{n}X_i\sum_{i=1}^{n}X_i\right]=\left\{(n-1)(1-\bar{\rho})\sigma^2\right\}^2=(n-1)^2(1-\bar{\rho})^2\sigma^4.$$

We need to expand $(n-1)^2(1-\bar{\rho})^2\sigma^4$. Now

$$(n-1)^2(1-\bar{\rho})^2\sigma^4=(n-1)^2\sigma^4\left\{1-\frac{1}{n(n-1)}\sum_{i\neq j}^{n(n-1)}\rho_{ij}\right\}^2$$

$$=(n-1)^2\sigma^4\left\{1-\frac{2}{n(n-1)}\sum_{i\neq j}^{n(n-1)}\rho_{ij}+\frac{1}{n^2(n-1)^2}\sum_{i\neq j}^{n(n-1)}\rho_{ij}\sum_{i\neq j}^{n(n-1)}\rho_{ij}\right\}$$

$$=(n-1)^2\sigma^4\left\{\begin{aligned}&1-\frac{2}{n(n-1)}\sum_{i\neq j}^{n(n-1)}\rho_{ij}\\&+\frac{1}{n^2(n-1)^2}\left\{2\sum_{i\neq j}^{n(n-1)}\rho_{ij}^2+4\sum_{i\neq j\neq k}^{n(n-1)(n-2)}\rho_{ij}\rho_{ik}+\sum_{i\neq j\neq k\neq l}^{n(n-1)(n-2(n-3)}\rho_{ij}\rho_{kl}\right\}\end{aligned}\right\},$$

which simplifies, giving

$$(n-1)^2(1-\bar{\rho})^2\sigma^4=\left\{n^2-2n+1\right\}\sigma^4+\left\{\frac{2}{n}-2\right\}\sigma^4\sum_{i\neq j}^{n(n-1)}\rho_{ij}$$

$$+\frac{2}{n^2}\sigma^4\sum_{i\neq j}^{n(n-1)}\rho_{ij}^2+\frac{4}{n^2}\sigma^4\sum_{i\neq j\neq k}^{n(n-1)(n-2)}\rho_{ij}\rho_{ik}+\frac{1}{n^2}\sigma^4\sum_{i\neq j\neq k\neq l}^{n(n-1)(n-2(n-3)}\rho_{ij}\rho_{kl}. \tag{D.21}$$

Substituting from equations (D.16), (D.17), (D.18), (D.19), (D.20), and (D.21) into equation (D.1) gives

$$Var\left[\sum_{i=1}^{n}(X_i-\bar{X})^2\right]$$

$$= E\left[\left\{\sum_{i=1}^{n} X_i^2 - \frac{1}{n}\sum_{i=1}^{n} X_i \sum_{i=1}^{n} X_i\right\}^2\right] - E^2\left[\sum_{i=1}^{n} X_i^2 - \frac{1}{n}\sum_{i=1}^{n} X_i \sum_{i=1}^{n} X_i\right].$$

$$= (n^2-1)\sigma^4 + \left\{-\frac{2}{n} - 2\right\}\sigma^4 \sum_{i\neq j}^{n(n-1)} \rho_{ij} + \left\{2 - \frac{4}{n} + \frac{6}{n^2}\right\}\sigma^4 \sum_{i\neq j}^{n(n-1)} \rho_{ij}^2$$

$$+ \left\{\frac{12}{n^2} - \frac{4}{n}\right\}\sigma^4 \sum_{i\neq j\neq k}^{n(n-1)(n-2)} \rho_{ij}\rho_{ik} + \frac{3}{n^2}\sigma^4 \sum_{i\neq j\neq k\neq l}^{n(n-1)(n-2(n-3)} \rho_{ij}\rho_{kl}$$

$$-(n^2 - 2n + 1)\sigma^4 - \left\{\frac{2}{n} - 2\right\}\sigma^4 \sum_{i\neq j}^{n(n-1)} \rho_{ij} - \frac{2}{n^2}\sigma^4 \sum_{i\neq j}^{n(n-1)} \rho_{ij}^2$$

$$-\frac{4}{n^2}\sigma^4 \sum_{i\neq j\neq k}^{n(n-1)(n-2)} \rho_{ij}\rho_{ik} - \frac{1}{n^2}\sigma^4 \sum_{i\neq j\neq k\neq l}^{n(n-1)(n-2(n-3)} \rho_{ij}\rho_{kl}$$

which equals

$$(2n-2)\sigma^4 - \frac{4}{n}\sigma^4 \sum_{i\neq j}^{n(n-1)} \rho_{ij} + \left\{2 - \frac{4}{n} + \frac{4}{n^2}\right\}\sigma^4 \sum_{i\neq j}^{n(n-1)} \rho_{ij}^2$$

$$+ \left\{\frac{8}{n^2} - \frac{4}{n}\right\}\sigma^4 \sum_{i\neq j\neq k}^{n(n-1)(n-2)} \rho_{ij}\rho_{ik} + \frac{2}{n^2}\sigma^4 \sum_{i\neq j\neq k\neq l}^{n(n-1)(n-2(n-3)} \rho_{ij}\rho_{kl}$$

which equals

$$2\sigma^4 \left\{ \begin{array}{l} (n-1) - \dfrac{2}{n}\displaystyle\sum_{i\neq j}^{n(n-1)} \rho_{ij} + \left\{1 - \dfrac{2}{n} + \dfrac{2}{n^2}\right\}\displaystyle\sum_{i\neq j}^{n(n-1)} \rho_{ij}^2 \\ + \left\{\dfrac{4}{n^2} - \dfrac{2}{n}\right\}\displaystyle\sum_{i\neq j\neq k}^{n(n-1)(n-2)} \rho_{ij}\rho_{ik} + \dfrac{1}{n^2}\displaystyle\sum_{i\neq j\neq k\neq l}^{n(n-1)(n-2(n-3)} \rho_{ij}\rho_{kl} \end{array} \right\}$$

which equals

$$2\sigma^4 \left\{ \begin{array}{l} (n-1) - \dfrac{2}{n}\displaystyle\sum_{i\neq j}^{n(n-1)} \rho_{ij} + \left\{\left\{1 - \dfrac{1}{n}\right\}^2 + \left\{\dfrac{1}{n}\right\}^2\right\}\displaystyle\sum_{i\neq j}^{n(n-1)} \rho_{ij}^2 \\ - \dfrac{2}{n}\left\{1 - \dfrac{2}{n}\right\}\displaystyle\sum_{i\neq j\neq k}^{n(n-1)(n-2)} \rho_{ij}\rho_{ik} + \left\{\dfrac{1}{n}\right\}^2\displaystyle\sum_{i\neq j\neq k\neq l}^{n(n-1)(n-2(n-3)} \rho_{ij}\rho_{kl} \end{array} \right\}. \qquad \text{(D.22)}$$

Now since

$$\sum_{i\neq j}^{n(n-1)}\rho_{ij}\sum_{i\neq j}^{n(n-1)}\rho_{ij}=2\sum_{i\neq j}^{n(n-1)}\rho_{ij}^2+4\sum_{i\neq j\neq k}^{n(n-1)(n-2)}\rho_{ij}\rho_{ik}+\sum_{i\neq j\neq k\neq l}^{n(n-1)(n-2(n-3)}\rho_{ij}\rho_{kl}\,,$$

we have the alternate expression for (D.22),

$$2\sigma^4\left\{\begin{array}{l}(n-1)-\dfrac{2}{n}\displaystyle\sum_{i\neq j}^{n(n-1)}\rho_{ij}+\left\{1-\dfrac{2}{n}\right\}\sum_{i\neq j}^{n(n-1)}\rho_{ij}^2\\ -\dfrac{2}{n}\displaystyle\sum_{i\neq j\neq k}^{n(n-1)(n-2)}\rho_{ij}\rho_{ik}+\left\{\dfrac{1}{n}\right\}^2\sum_{i\neq j}^{n(n-1)}\rho_{ij}\sum_{i\neq j}^{n(n-1)}\rho_{ij}\end{array}\right\}. \tag{D.23}$$

The expression in (D.22) or (D.23) is $Var\left[\sum_{i=1}^{n}(X_i-\bar{X})^2\right]$. We denote the sum of the terms in the braces of (D.22) or (D.23) as $trace\mathbf{CRCR}$, and so

$$Var\left[\sum_{i=1}^{n}(X_i-\bar{X})^2\right]=2\sigma^4 trace\mathbf{CRCR}. \tag{D.24}$$

This notation is not accidental. Let $\mathbf{x}$ be a multivariate normal random n-vector with mean vector μ with identical coordinates, correlation matrix $\mathbf{R}$, and covariance matrix $\sigma^2\mathbf{R}$. Let $\mathbf{C}=\mathbf{I}-\frac{1}{n}\mathbf{1}\mathbf{1}'$ be the centering matrix, that is

$$\mathbf{Cx}=\mathbf{x}-\bar{\mathbf{x}}=(X_1-\bar{X},X_2-\bar{X},\cdots,X_n-\bar{X})'$$

and

$$\mathbf{x}'\mathbf{Cx}=\mathbf{x}'\mathbf{CCx}=(\mathbf{x}-\bar{\mathbf{x}})'(\mathbf{x}-\bar{\mathbf{x}})=\sum_{i=1}^{n}(X_i-\bar{X})^2.$$

Now use the fact, from Searle (1982), that

$$\begin{aligned}Var[\mathbf{x}'\mathbf{Cx}]&=2trace\left[(\mathbf{C}\sigma^2\mathbf{R})(\mathbf{C}\sigma^2\mathbf{R})\right]+4\mu'\mathbf{C}(\sigma^2\mathbf{R})\mathbf{C}\mu\\&=2\sigma^4 trace\mathbf{CRCR}+0\\&=2\sigma^4 trace\mathbf{CRCR},\end{aligned}$$

and so

$$Var\left[\sum_{i=1}^{n}(X_i - \bar{X})^2\right] = Var[\mathbf{x}'\mathbf{C}\mathbf{x}] = 2\sigma^4 trace\mathbf{CRCR},$$

that is, the expression in (D.22) or (D.23) is $Var\left[\sum_{i=1}^{n}(X_i - \bar{X})^2\right]$ and the sum of the terms in the braces of (D.22) or (D.23) is $trace\mathbf{CRCR}$.

Note that in the case of uncorrelated characteristics $\{X_i\}_n$,

$$Var\left[\sum_{i=1}^{n}(X_i - \bar{X})^2\right] = 2\sigma^4 trace\mathbf{CRCR}$$

$$= 2\sigma^4 trace\mathbf{CICI} = 2\sigma^4 trace\mathbf{CC}$$

$$= 2\sigma^4 trace\mathbf{C} = 2\sigma^4(n-1),$$

since $\mathbf{C} = \mathbf{I} - \frac{1}{n}\mathbf{1}\mathbf{1}'$ is idempotent with its rank equal to its trace, each equaling $(n - 1)$.

Therefore,

$$Var[S^2] = Var\left[\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2\right] = \frac{1}{(n-1)^2}Var\left[\sum_{i=1}^{n}(X_i - \bar{X})^2\right]$$

$$= \frac{1}{(n-1)^2}2\sigma^4(n-1)$$

$$= \frac{2\sigma^4}{n-1},$$

which is consistent with known normal theory.

VITA

Lawrence Lee Magee was born in New Orleans, Louisiana, in 1948. He graduated from Neville High School, Monroe, Louisiana, in 1966. He attended Louisiana State University in Baton Rouge, Louisiana, earning the bachelor of science degree in mathematics in 1971 and the master of science degree in quantitative methods in 1977. In 1985, he earned the master of science degree in business from the University of Wisconsin, Madison, Wisconsin. He is currently an instructor at Louisiana State University in Baton Rouge, and a candidate for the degree of Doctor of Philosophy, to be conferred in December, 1998. He has also served as instructor at the University of Wisconsin, Madison, Wisconsin, and at Southwest Texas State University, San Marcos, Texas. His research and teaching interests include time series analysis, Bayesian decision analysis, and experimental design in quality management. He lives in Zachary, Louisiana.

DOCTORAL EXAMINATION AND DISSERTATION REPORT

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Approved:

Helmut Schneider

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Date of Examination:

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